Semistable reduction for overconvergent F-isocrystals, III: Local semistable reduction at monomial valuations

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Abstract

We resolve the local semistable reduction problem for overconvergent F-isocrystals at monomial valuations (Abhyankar valuations of height 1 and residue transcendence degree 0). We first introduce a higher-dimensional analogue of the generic radius of convergence for a p-adic differential module, which obeys a convexity property. We then combine this convexity property with a form of the p-adic local monodromy theorem for so-called fake annuli.

Contents

1 Introduction

This paper is the third of a series, preceded by [16, 17]. The goal of the series is to prove a "semistable reduction" theorem for overconvergent F-isocrystals, a class of p-adic analytic objects associated to schemes of finite type over a field of characteristic $p > 0$. Such a theorem is expected to have consequences for the theory of rigid cohomology, in which overconvergent F-isocrystals play the role of coefficient objects.

In [16], it was shown that the problem of extending an overconvergent isocrystal on a variety X to a log-isocrystal on a larger variety $\overline{X}$ is governed by the triviality of a sort of local monodromy along components of the complement of X. In [17], it was shown that the problem can be localized on the space of valuations on the function field of the given variety. In this paper, we solve the local semistable reduction problem at monomial valuations (Abhyankar valuations of height 1 and residue transcendence degree 0).

The context of this result (including a complex analogue) and a description of potential applications is already given in the introduction of [16], so we will not repeat it here. Instead, we devote the remainder of this introduction to an overview of the results specific to this paper, and a survey of the structure of the various sections of the paper.

1.1 Local semistable reduction

The problem of (global) semistable reduction is to show that an overconvergent F-isocrystal on a nonproper k-variety can be extended to a log-isocrystal with nilpotent residues on a proper k-variety after passing to a suitable generically finite cover. In [16], it was shown that existence of a log-extension on a smooth pair (X, D) can be checked generically along each component of D. One can ensure the existence of the log-extension along the proper transform of any given component, using the p-adic local monodromy theorem of André [2], Mebkhout [23], and this author [11]; however, when passing to the generically finite cover (and making sure it is smooth by applying de Jong's alterations theorem [7]), one typically introduces exceptional components along which one has not achieved any control of local monodromy. See Example A.2.1 for an explicit example of this phenomenon.

The main addition of [17] was to show that the problem of controlling exceptional components can be localized within the Riemann-Zariski space of valuations of the function field of the variety. Moreover, the resulting problem of local semistable reduction can be reduced to a lower-dimensional case whenever one is working in neighborhoods of a valuation which is composite (of height greater than 1), or which has a residue field with positive transcendence degree over the base field.

1.2 Local monodromy at monomial valuations

Monomial valuations on an n-dimensional variety can be described as follows: in suitable local coordinates $x_1, \dots, x_n$, they are determined by the fact that $v(x_1), \dots, v(x_n)$ are linearly independent over $\mathbb{Q}$. On one hand, such valuations are particularly easy to describe, so one expects to have an easier time working with them than with other valuations. On the other hand, they form a subset of the Riemann-Zariski space which, in some sense that we will not make precise here, is rather large. (A related statement is that the set of Abhyankar places of a finitely generated field extension is dense in the Riemann-Zariski space under the patch topology [22, Corollary 2].)

In order to understand the structure of isocrystals in a neighborhood of a monomial valuation, it is helpful to make some analysis at the valuation itself. This is the content of the paper [14], which proves an analogue of the p-adic local monodromy theorem for differential equations on a so-called *fake annulus* inside a higher-dimensional affine space. It then makes sense to consider the (semisimplified) local monodromy representations attached to overconvergent F-isocrystals not just at divisorial valuations, but also at monomial valuations.

One is then led to ask how local monodromy varies as one varies the monomial valuation, e.g., by varying $v(x_1), \dots, v(x_n)$. We give a tangible answer to this question by defining a higher-dimensional analogue of the generic radius of convergence, as considered by Christol-Dwork [4]. This gives a numerical invariant which in the one-dimensional case computes the highest ramification break of the local monodromy representation, as in the work of André, Christol-Mebkhout, Crew, Matsuda, Tsuzuki, et al. (See [12, § 5] for an exposition.) This number is shown to be a *convex* function in $v(x_1), \dots, v(x_n)$ by the Hadamard three circles lemma in rigid geometry. (One can similarly construct an invariant that generalizes the full

Swan conductor in the one-dimensional case; see [15] for the beginning of this story.)

It is worth noting that this study has an interesting analogue over the complex numbers, in the investigation of the Stokes phenomenon conducted by Sabbah [29]. This concerns irregular connections on complex surfaces, and (echoing an analogy already seen in the one-dimensional situation) the variational behavior of irregularity along divisors is apparently quite similar to that of the invariant we consider.

Given what we have just described, we prove local semistable reduction at a monomial valuation as follows. Using the p-adic local monodromy theorem for fake annuli, we can force the highest ramification break at the valuation itself to be zero. Then the convexity of the highest break function implies that at certain nearby divisorial valuations, the highest break is also forced to be zero, which forces the isocrystal to be unipotent there also.

1.3 Structure of the paper

We conclude this introduction with a summary of the structure of the paper.

In Section 2, we derive some properties of convex functions. The most important of these is a result which we were unable to find in the literature (Theorem 2.3.2), which shows that a convex function whose values have the divisibility properties of a piecewise affine function with integral coefficients must in fact be such a function.

In Section 3, we recall the relationship between Newton polygons and norms of differential operators, as developed by Christol-Dwork, Robba, Young, et al.

In Section 4, we define our higher-dimensional analogue of generic radius of convergence, and gather its key properties.

In Section 5, we introduce a form of the p-adic local monodromy theorem covering so-called fake annuli. We then assert some related results, notably the relationship between wild ramification and generic radius of convergence for differential equations on p-adic curves.

In Section 6, we develop some properties of monomial valuations, then prove local semistable reduction at a monomial valuation using the log-concavity of generic radius of convergence.

In the Appendix, we describe two examples of semistable reduction. One illustrates that one cannot insist on using a finite cover of a fixed compactification, rather than an alteration (as promised in the introduction of [16]). The other illustrates that even if one starts with a good compactification, one cannot achieve semistable reduction by doing so just for the divisors visible in that compactification (as promised above).

Notation 1.3.1. We retain the basic notations of [16, 17]. In particular, k will always denote a field of characteristic $p > 0$, K will denote a complete discretely valued field of characteristic zero with residue field k, equipped with an continuous endomorphism σ_K lifting the q-power Frobenius for some power q of p, and $\mathfrak{o}_K$ will denote the ring of integers of K.

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2 Some properties of convex functions

This section is completely elementary; it consists of some basic properties of convex functions on subsets of $\mathbb{R}^n$, which we will use later to study variation of the highest ramification break as a function of a valuation on a variety. We initially follow [28] for notation and terminology.

2.1 Convex functions

In the study of convex functions, as in [28], it is customary to use a slightly different setup than one might expect.

Definition 2.1.1. Denote $\mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$. A function $f : \mathbb{R}^n \to \mathbb{R}_\infty$ is *convex* if for any $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$,

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y).$$

Equivalently, f is convex if and only if the *epigraph* of f, defined as

$$\mathrm{epi}(f) = \{(x_1, \dots, x_n, y) \in \mathbb{R}^{n+1} : y \geq f(x_1, \dots, x_n)\},$$

is a convex set.

Definition 2.1.2. If U is a convex subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ is a function, we say f is *convex* if the function $g : \mathbb{R}^n \to \mathbb{R}_\infty$ defined by

$$g(x) = \begin{cases} f(x) & x \in U \\ +\infty & x \notin U \end{cases}$$

is convex in the sense of Definition 2.1.1. Conversely, for $g : \mathbb{R}^n \to \mathbb{R}_\infty$ a convex function, we define the *essential domain* of g to be

$$\mathrm{dom}(g) = \{x \in \mathbb{R}^n : g(x) < +\infty\};$$

then the restriction of g to $\mathrm{dom}(g)$ is a convex function in the sense just described. Write $\mathrm{intdom}(g)$ for the interior of $\mathrm{dom}(g)$; then g is continuous on $\mathrm{intdom}(g)$ [28, Theorem 10.1].

Definition 2.1.3. For $C \subseteq \mathbb{R}^n$, define the *indicator function* $\delta_C : \mathbb{R}^n \to \mathbb{R}_\infty$ by

$$\delta_C(x) = \begin{cases} 0 & x \in C \\ +\infty & x \notin C; \end{cases}$$

then δ_C is a convex function if and only if C is a convex set.

Definition 2.1.4. An *affine functional* is a map $\lambda : \mathbb{R}^n \to \mathbb{R}$ of the form $\lambda(x_1, \dots, x_n) = a_1x_1 + \cdots + a_nx_n + b$ for some $a_1, \dots, a_n, b \in \mathbb{R}$. A *generalized affine functional* is a map $\lambda : \mathbb{R}^n \to \mathbb{R}_\infty$ which is either an affine functional, or an affine functional plus the indicator function of a *closed halfspace*, i.e., a set of the form

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : a_1x_1 + \cdots + a_nx_n \leq b\}$$

with $a_1, \dots, a_n$ not all zero. If $a_1, \dots, a_n, b \in \mathbb{Z}$ (in both places, if working in the generalized case), we say λ is an *integral* (generalized) affine functional.

Lemma 2.1.5. *Let $f : \mathbb{R}^n \to \mathbb{R}_\infty$ be a convex function. Then for any $m \in \mathbb{R}$, the function $g_m : \mathbb{R}^{n-1} \to \mathbb{R}_\infty$ defined by*

$$g_m(x_1, \dots, x_{n-1}) = \inf_{x_n \in \mathbb{R}} \{f(x_1, \dots, x_n) - mx_n\}$$

is convex.

Proof. Since $f(x_1, \dots, x_n) - mx_n$ is again a convex function of $x_1, \dots, x_n$, it suffices to consider the case $m = 0$. Given $x_{1,1}, \dots, x_{1,n-1}, x_{2,1}, \dots, x_{2,n-1} \in \mathbb{R}$ and $\epsilon > 0$, choose $x_{1,n}, x_{2,n} \in \mathbb{R}$ such that

$$f(x_{i,1}, \dots, x_{i,n}) \leq g_0(x_{i,1}, \dots, x_{i,n-1}) + \epsilon \qquad (i = 1, 2).$$

For $t \in [0, 1]$, put $x_{3,j} = tx_{1,j} + (1-t)x_{2,j}$ for $j = 1, \dots, n$. Write $x_i = (x_{i,1}, \dots, x_{i,n})$ and $x'_i = (x_{i,1}, \dots, x_{i,n-1})$ for $i = 1, 2, 3$. Then

$$tg_0(x'_1) + (1-t)g_0(x'_2) + 2\epsilon \geq tf(x_1) + (1-t)f(x_2) \geq f(x_3) \geq g_0(x'_3).$$

Taking ϵ arbitrarily small, we deduce the convexity of g_0. □

2.2 Internally polyhedral functions

Definition 2.2.1. For $f : \mathbb{R}^n \to \mathbb{R}_\infty$ a convex function, a *domain of affinity* is a subset U of $\mathbb{R}^n$ with nonempty interior on which f agrees with an affine functional λ. The nonempty interior condition ensures that λ is uniquely determined; we call it the *ambient functional* on U.

Remark 2.2.2. Note that if λ is an ambient functional on some domain of affinity for f, then the graph of λ is a supporting hyperplane for the epigraph of f, and so $f(x) \geq \lambda(x)$ for all $x \in \mathbb{R}^n$.

Definition 2.2.3. A function $f : \mathbb{R}^n \to \mathbb{R}_\infty$ is *polyhedral* if it has the form

$$f(x) = \max\{\lambda_1(x), \dots, \lambda_m(x)\} \tag{2.2.3.1}$$

for some generalized affine functionals $\lambda_1, \dots, \lambda_m : \mathbb{R}^n \to \mathbb{R}_\infty$. We say f is *integral polyhedral* if the λ_i can be taken to be integral; we say a set C is *rational polyhedral* if the indicator function δ_C is integral polyhedral. We say f is *internally (integral) polyhedral* if for each bounded (rational) polyhedral set $C \subseteq \operatorname{intdom}(f)$, $f + \delta_C$ is (integral) polyhedral.

Remark 2.2.4. It may look a bit strange to say that C is *rational* polyhedral if δ_C is *integral* polyhedral. The point is that to get what one would properly call an "integral polyhedral set", i.e., the convex hull of a finite subset of $\mathbb{Z}^n$, we would have to force the λ_i in (2.2.3.1) to have the form $a_1x_1 + \cdots + a_nx_n + b$ in which $a_1, \dots, a_n, b \in \mathbb{Z}$ but $a_1, \dots, a_n$ are additionally constrained to be coprime.

Remark 2.2.5. The condition that a function f be internally polyhedral is more permissive than the condition that it be *locally polyhedral*, in the sense of, e.g., [10, §15]. To say that f is locally polyhedral means that for every bounded polyhedral set C meeting $\operatorname{dom}(f)$, $f + \delta_C$ is polyhedral. To see the difference, note that the functions $f : \mathbb{R} \to \mathbb{R}_\infty$ given by

$$f(x) = \begin{cases} +\infty & x \leq 0 \\ 2N - N(N+1)x & x \in [1/(N+1), 1/N], N \in \mathbb{Z}_{>0} \\ 0 & x \geq 1 \end{cases}$$

and by

$$f(x) = \begin{cases} +\infty & x < 0 \\ 1 & x = 0 \\ 0 & x > 0 \end{cases}$$

are internally integral polyhedral but not locally polyhedral.

Lemma 2.2.6. *(a) Let $f : \mathbb{R}^n \to \mathbb{R}_\infty$ be a convex function such that* $\operatorname{dom}(f)$ *is polyhedral with nonempty interior. Then f is polyhedral if and only if* $\operatorname{dom}(f)$ *is covered by finitely many domains of affinity for f.*

(b) Let $f : \mathbb{R}^n \to \mathbb{R}_\infty$ be a convex function. Then f is internally polyhedral if and only if $\operatorname{intdom}(f)$ *is covered by (possibly infinitely many) domains of affinity for f.*

Proof. The "only if" implication is evident in both cases, so we focus on the "if" implications.

To prove (a), put $C = \operatorname{dom}(f)$; since C is polyhedral, we can write

$$\delta_C(x) = \max\{\mu_1(x), \dots, \mu_r(x)\}$$

where each μ_j is the indicator function of a closed halfspace. If $\operatorname{dom}(f)$ is covered by domains of affinity $U_1, \dots, U_m$ with ambient functionals $\lambda_1, \dots, \lambda_m$, then by Remark 2.2.2, we have

$$f(x) = \max_{i,j}\{\lambda_i(x) + \mu_j(x)\},$$

so f is polyhedral.

To prove (b), let C be any bounded rational polyhedral subset of $\operatorname{intdom}(f)$. Since C is compact, it is covered by finitely many domains of affinity for f; hence (a) implies that $f+\delta_C$ is polyhedral. Since $\operatorname{intdom}(f)$ is the union of its bounded rational polyhedral subsets, this proves the claim. □

Corollary 2.2.7. *Let $f : \mathbb{R}^n \to \mathbb{R}_\infty$ be a convex function. Then f is internally integral polyhedral if and only if there exists a (possibly infinite) subset S of $\mathbb{Z}^n$ and a function $b : S \to \mathbb{Z}$ such that*

$$f(x) = \sup_{s\in S}\{s_1x_1 + \cdots + s_nx_n + b(s)\} \qquad (x \in \operatorname{intdom}(f)). \tag{2.2.7.1}$$

Moreover, in this case the supremum in (2.2.7.1) *is always achieved.*

Proof. If f is internally integral polyhedral, we choose S and b so that $s_1x_1+\cdots+s_nx_n+b(s)$ runs over the ambient functionals on the domains of affinity of f; then Remark 2.2.2 implies (2.2.7.1) with the supremum being achieved. Conversely, suppose S and b exist. Pick $x \in \operatorname{intdom}(f)$, and then choose $\epsilon > 0$ such that the box $B = \prod_{i=1}^n [x_i - 2\epsilon, x_i + 2\epsilon]$ is contained in $\operatorname{intdom}(f)$. Put $B' = \prod_{i=1}^n [x_i - \epsilon, x_i + \epsilon]$. Let U and L be the supremum and infimum, respectively, of f on B (which exist because f is continuous on $\operatorname{intdom}(f)$). If $s_1 > (U - L)/\epsilon$, then for $y = (y_1, \dots, y_n) \in B'$,

$$\begin{aligned}
f(y) &\geq L \\
&> U - s_1\epsilon \\
&\geq U - s_1(x_1 + 2\epsilon - y_1) \\
&= U - (s_1(x_1 + 2\epsilon) + s_2y_2 + \cdots + s_ny_n + b(s)) + (s_1y_1 + \cdots + s_ny_n + b(s)) \\
&\geq U - f(x_1 + 2\epsilon, y_2, \dots, y_n) + s_1y_1 + \cdots + s_ny_n + b(s) \\
&\geq s_1y_1 + \cdots + s_ny_n + b(s).
\end{aligned}$$

That is, for all $x \in B'$, any term in (2.2.7.1) for an s with $s_1 > (U - L)/\epsilon$ can be omitted without changing the supremum. Similarly, we can omit all s for which $|s_i| > (U - L)/\epsilon$ for $i \in \{1, \dots, n\}$. Consequently, we can compute the supremum in (2.2.7.1) using only finitely many affine functionals, and so $f + \delta_C$ is integral polyhedral for any rational polyhedron $C \subseteq B'$.

Consequently, the point $x \in \operatorname{intdom}(f)$ admits a neighborhood contained in a union of finitely many domains of affinity for f corresponding to integral affine functionals. Since x was arbitrary, we deduce that all of $\operatorname{intdom}(f)$ can be covered by domains of affinity for f whose ambient functionals are integral. By Lemma 2.2.6, f is internally integral polyhedral, as desired. □

2.3 Integral values and integral polyhedral functions

The key result in this section (Theorem 2.3.2) asserts that the fact that a convex function is internally integral polyhedral can be observed from its values at rational n-tuples.

Lemma 2.3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}_\infty$ be a convex function such that*

$$f(x_1, \dots, x_n) \in (\mathbb{Z} + \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n) \cup \{+\infty\} \qquad (x_1, \dots, x_n \in \mathbb{Q}). \tag{2.3.1.1}$$

Then for any $x_1, \dots, x_{n-1} \in \mathbb{Q}$, the function $g : \mathbb{R} \to \mathbb{R}_\infty$ given by $g(x) = f(x_1, \dots, x_{n-1}, x)$ is internally polyhedral, and on each domain of affinity of g, we have $g(x) = mx + b$ for some $m \in \mathbb{Z}$ and some $b \in \mathbb{Z} + \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_{n-1}$.

Proof. Fix $(x_1, \dots, x_{n-1}) \in \mathbb{Q}^n$. We may assume that $\mathrm{intdom}(g)$ is nonempty, as otherwise the claim is vacuously true; choose $x_n \in \mathrm{intdom}(g)$. Let d be the least common denominator of $x_1, \dots, x_n$, so that

$$\frac{1}{d}\mathbb{Z} = \mathbb{Z} + \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n.$$

For N any sufficiently large positive integer, we have

$$\frac{f(x_1, \dots, x_{n-1}, x_n + 1/(dN)) - f(x_1, \dots, x_n)}{1/(dN)} \in dN\left(\mathbb{Z} + \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n + \frac{1}{dN}\mathbb{Z}\right) = \mathbb{Z}.$$

As $N \to \infty$, this difference quotient runs through a sequence of integers which is nonincreasing and bounded below (because g is convex and $x_n \in \mathrm{intdom}(g)$). Thus the quotient stabilizes for N large. By convexity, the function g must be affine with integral slope in a one-sided neighborhood of x_n; since $\mathbb{Q}$ is dense in $\mathbb{R}$, the closed intervals on which g is affine with integral slope cover the interior of the essential domain. By Lemma 2.2.6, g is internally polyhedral.

Let d' be the least common denominator of $x_1, \dots, x_{n-1}$. On any domain of affinity for g, we can write $g(x) = mx + b$ for some $m \in \mathbb{Z}$. In this domain, we can find y_1, y_2 such that when we write $y_i = r_i/s_i$ in lowest terms, we have d', s_1, s_2 coprime in pairs. From (2.3.1.1), we have $g(y_i) = m(r_i/s_i) + b \in \frac{1}{d's_i}\mathbb{Z}$, implying $d's_i b \in \mathbb{Z}$. Since this holds for both $i = 1$ and $i = 2$, we find $d'b \in \mathbb{Z}$. □

Theorem 2.3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}_\infty$ be a convex function such that*

$$f(x_1, \dots, x_n) \in (\mathbb{Z} + \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n) \cup \{+\infty\} \qquad (x_1, \dots, x_n \in \mathbb{Q}).$$

Then f is internally integral polyhedral.

Proof. We proceed by induction on n, the case $n = 1$ being solved by Lemma 2.3.1. Write for brevity $x = (x_1, \dots, x_n)$ and $x' = (x_1, \dots, x_{n-1})$. For $m \in \mathbb{Z}$, define

$$g_m(x') = \inf_{x \in \mathbb{R}}\{f(x_1, \dots, x_{n-1}, x) - mx\};$$

by Lemma 2.1.5, g_m is a convex function on $\mathbb{R}^{n-1}$. By Lemma 2.3.1, for $x' \in \mathbb{Q}^{n-1} \cap \mathrm{intdom}(g_m)$, $g_m(x') \in \mathbb{Z} + \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_{n-1}$. We may thus apply the induction hypothesis to deduce that g_m is internally integral polyhedral.

By one direction of Corollary 2.2.7, we can construct sets $S_m \subseteq \mathbb{Z}^{n-1}$ and functions $b_m : S_m \to \mathbb{Z}$ such that

$$g_m(x') = \sup_{s \in S_m} \{s_1 x_1 + \cdots + s_{n-1} x_{n-1} + b_m(s)\} \qquad (x' \in \operatorname{intdom}(g_m)).$$

By Lemma 2.3.1, we know that for $x \in \mathbb{Q}^n$,

$$f(x) = \sup_{m \in \mathbb{Z}} \{g_m(x_1, \dots, x_{n-1}) + m x_n\},$$

so we conclude that

$$f(x) = \sup\{s_1 x_1 + \cdots + s_{n-1} x_{n-1} + m x_n + b_m(s)\} \tag{2.3.2.1}$$

for all $x \in \mathbb{Q}^n \cap \operatorname{intdom}(f)$, with the supremum running over $m \in \mathbb{Z}$ and $s = (s_1, \dots, s_{n-1}) \in S_m$. Since both sides of (2.3.2.1) represent convex functions on $\operatorname{intdom}(f)$ and they agree on a dense subset thereof, we may invoke the continuity of convex functions to deduce that (2.3.2.1) holds in fact for all $x \in \operatorname{intdom}(f)$. By the other direction of Corollary 2.2.7, f is internally integral polyhedral, as desired. □

2.4 Extension to rational polyhedral sets

Although we will not use it in this paper, we note for future reference a slight strengthening of Theorem 2.3.2.

Lemma 2.4.1. *Let C be a bounded rational polyhedral subset of $\mathbb{R}^n$, and let $v \in \mathbb{Q}^n$ be a vertex of C. Let $f : \mathbb{R}^n \to \mathbb{R}_\infty$ be a convex function with $f(v) < \infty$, and let T_v be the set of integral affine functionals λ which achieve their maximum on C at v, and which agree with f on some domain of affinity meeting C. Then T_v is finite.*

Proof. Let $S \subset \mathbb{Z}^n$ be the set of n-tuples for which

$$\max_{x \in C} \{s_1 x_1 + \cdots + s_n x_n\} = s_1 v_1 + \cdots + s_n v_n.$$

Then S is the intersection of $\mathbb{Z}^n$ with a strictly convex rational polyhedral cone, and so is isomorphic to the intersection of $\mathbb{Z}^n_{\geq 0}$ with a sublattice of $\mathbb{Z}^n$ of finite index. Consequently, S is well partially ordered, that is, any infinite sequence of S contains an infinite nondecreasing subsequence.

Suppose that T_v is infinite. For $\lambda \in T_v$, write

$$\lambda(x) = s_1 x_1 + \cdots + s_n x_n + b$$

with $s = s(\lambda) \in S$ and $b = b(\lambda)$. Note that no two $\lambda \in T_v$ can have the same $s(\lambda)$, by Remark 2.2.2. By the above, we can choose $\lambda^{(1)}, \lambda^{(2)}, \dots \in T_v$ so that the corresponding $s^{(i)} = s(\lambda^{(i)})$ form an infinite increasing sequence.

For any x in a domain of affinity for f on which f agrees with $\lambda^{(i)}$, we must have $\lambda^{(i)}(x) = f(x) \geq \lambda^{(i-1)}(x)$ because f is convex; moreover, because $\lambda^{(i)} \neq \lambda^{(i-1)}$, we can choose x so that the inequality is strict. Since $s^{(i)} - s^{(i-1)} \in S$ by construction, we have

$$\lambda^{(i)}(v) - \lambda^{(i-1)}(v) \geq \lambda^{(i)}(x) - \lambda^{(i-1)}(x) > 0.$$

However, for all i, we have $f(v) \geq \lambda^{(i)}(v)$. Consequently, the $\lambda^{(i)}(v)$ form a strictly increasing, bounded above sequence with values in the discrete subset $\mathbb{Z} + \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ of $\mathbb{R}$. This is impossible, yielding the desired contradiction. □

Theorem 2.4.2. *Let C be a bounded rational polyhedral subset of $\mathbb{R}^n$. Then a continuous convex function $f : C \to \mathbb{R}$ is integral polyhedral if and only if*

$$f(x) \in \mathbb{Z} + \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n \qquad (x \in C \cap \mathbb{Q}^n). \tag{2.4.2.1}$$

Proof. If f is integral polyhedral, then (2.4.2.1) is clear. Conversely, assume (2.4.2.1); by Theorem 2.3.2, f is internally integral polyhedral.

Let T be the set of integral affine functionals λ which agree with f on some domain of affinity. For v a vertex of C, let T_v be the set of $\lambda \in T$ which achieve their maximum on C at v. By Lemma 2.4.1, each T_v is finite; since C has only finitely many vertices, and T is the union of the T_v, T must be finite. By Lemma 2.2.6, f is integral polyhedral, as desired. □

3 Differential equations and Newton polygons

In this section, we review the relationship between differential equations over complete valued fields and Newton polygons. The analysis here draws from Young [32], Christol-Dwork [4], and particularly Robba [27].

Hypothesis 3.0.1. Until §3.4, let F denote a *valued (nontrivial) differential field* of characteristic zero. That is, F is a field equipped with a nonzero derivation $\partial : F \to F$, and with a nonarchimedean absolute value $|\cdot|$; we write $v(\cdot) = -\log|\cdot|$ for the corresponding valuation. We will later require that F be complete (starting in §3.2).

3.1 Valued differential fields and twisted polynomials

Definition 3.1.1. For T a bounded linear operator on a normed vector space V, the *operator norm* of T, denoted $|T|_V$, is the infimum of those $c \in \mathbb{R}_{\geq 0}$ for which $|T(x)| \leq c|x|$ for all $x \in V$. For $m, n \in \mathbb{Z}_{\geq 0}$, we have the evident inequality

$$|T^{m+n}|_V \leq |T^m|_V |T^n|_V.$$

By taking logarithms, we arrive at the situation of Fekete's lemma: if $\{a_n\}_{n=1}^\infty$ is a sequence of reals with $a_{m+n} \leq a_m + a_n$ for all m, n, then the sequence $\{a_n/n\}_{n=1}^\infty$ either converges to

its infimum or diverges to $-\infty$ [25, Part 1, Problem 98]. We may thus define the *spectral norm* of T as

$$|T|_{V,\mathrm{sp}} = \lim_{n\to\infty} |T^n|_V^{1/n} = \inf_n \{|T^n|_V^{1/n}\};$$

it depends only on the equivalence class of the norm on V. In particular, we will apply this notation with $T = \partial$ acting on F (as a vector space over the subfield killed by ∂); put $r_0 = -\log|\partial|_F$.

Definition 3.1.2. Let $F\{T\}$ denote the twisted polynomial ring over F in the sense of Ore [24], so that for $x \in F$, $Tx = xT + \partial(x)$. By the Leibniz rule, for $x \in F$,

$$T^n x = \sum_{i=0}^{n} \binom{n}{i} \partial^{n-i}(x) T^i.$$

The twisted polynomial ring admits division with remainder on both sides, so the Euclidean algorithm applies to show that left ideals and right ideals are all principal (again, see [24]).

Remark 3.1.3. Note that the opposite ring of $F\{T\}$ is also a twisted polynomial ring, for the difference field F' with the same underlying field as F, but with derivation $-\partial$. The passage to the opposite ring corresponds, in the classical language of differential equations, to replacing a differential operator with its adjoint.

Definition 3.1.4. For $P = \sum c_n T^n \in F\{T\}$, define the *Newton polygon* of P as the lower convex hull of the set

$$\{(-n, v(c_n)) : n \in \mathbb{Z}_{\geq 0}, c_n \neq 0\}.$$

Define the *multiplicity* of a real number r (as a slope of P) as the width of the segment of the Newton polygon of slope r, or 0 if there is no such segment. For $r \in \mathbb{R}$, define

$$v_r(P) = \min_n \{rn + v(c_n)\};$$

this is the y-intercept of the supporting line of the Newton polygon of slope r. Note that for P fixed, $v_r(P)$ is a continuous function of r.

As originally observed by Robba [27, §1], this Newton polygon behaves like its counterpart for untwisted polynomials, but only for slopes which are not too large.

Lemma 3.1.5. *For $P, Q \in F\{T\}$ and $r \leq r_0$, we have $v_r(PQ) = v_r(P) + v_r(Q)$.*

Proof. Write $P = \sum_i a_i T^i$ and $Q = \sum_j b_j T^j$; then

$$PQ = \sum_k \left(\sum_{i+j=k} \sum_{h \geq 0} \binom{i+h}{h} a_{i+h} \partial^h(b_j) \right) T^k,$$

and hence

$$\begin{aligned}
v_r(PQ) &\geq \min_{h,i,j}\{v(a_{i+h}) + v(b_j) + r(i+j) - \log|\partial^h|_F\} \\
&\geq \min_{h,i,j}\{v(a_{i+h}) + v(b_j) + r(i+j) + hr_0\} \\
&\geq \min_{h,i,j}\{v(a_{i+h}) + v(b_j) + r(i+h+j)\}.
\end{aligned} \tag{3.1.5.1}$$

This immediately yields $v_r(PQ) \geq v_r(P) + v_r(Q)$. To establish equality for $r < r_0$, let i_0 and j_0 be the smallest values of i and j which minimize $ri + v(a_i)$ and $rj + v(b_j)$, respectively; then (3.1.5.1) achieves its minimum for $h = 0, i = i_0, j = j_0$ but not for any other h, i, j with $i + j = i_0 + j_0$. Hence $v_r(PQ) = v_r(P) + v_r(Q)$; equality for $r = r_0$ follows by continuity. (Compare [27, Proposition 1.6(2)].) ☐

Corollary 3.1.6. *For $P, Q \in F\{T\}$ and $r < r_0$, the multiplicity of r as a slope of PQ is the sum of its multiplicities as a slope of P and of Q.*

The moral here is that when one is only looking at phenomena in slopes less than r_0, one does not see the difference between twisted and untwisted polynomials. For instance, here is an explicit instance of this conclusion modeled on [4, Lemme 1.4] (compare also [27, Proposition 1.6(1)]).

Lemma 3.1.7. *For $r \leq r_0$, let $Q = U^d + \sum_{i=0}^{d-1} a_i U^i \in F[U]$ be a polynomial with all slopes at least r. (Here $F[U]$ denotes the untwisted polynomial ring.) Put $W = F[U]/F[U]Q$ as an F-vector space with norm $|\sum_{i=0}^{d-1} c_i U^i| = \max\{|c_i| e^{-ri}\}$. Let U act on W by left multiplication, and let ∂ act coordinatewise with respect to the basis $1, U, \dots, U^{d-1}$. Then*

$$|(U + \partial)^n|_W \leq e^{-rn} \qquad \textit{for all } n \in \mathbb{Z}_{\geq 0};$$

moreover, equality holds in case $r < r_0$ and Q has all slopes equal to r.

Proof. Rewrite the slope hypothesis as $|a_i|_F \leq e^{-r(d-i)}$ for $i = 0, \dots, d-1$; then clearly $|U^n|_W \leq |U|_W^n = e^{-rn}$, so $|(U + \partial)^n - U^n|_W \leq e^{-r(n-1)}|\partial|_F$. This yields all of the claims. ☐

3.2 Splitting over a complete field

For F complete, we obtain Robba's analogue for differential operators [27, Théorème 2.4] of Hensel's lemma for an untwisted polynomial over a complete nonarchimedean field.

Hypothesis 3.2.1. Throughout this subsection and the next, assume that F is complete for its norm.

Proposition 3.2.2. *Fix $r < r_0$ and $m \in \mathbb{Z}_{\geq 0}$. Let $R \in F\{T\}$ be a twisted polynomial such that $v_r(R - T^m) > v_r(T^m)$. Then R can be factored uniquely as PQ, where $P \in F\{T\}$ has degree $\deg(R) - m$ and all slopes less than r, $Q \in F\{T\}$ is monic of degree m and has all slopes greater than r, $v_r(P - 1) > 0$, and $v_r(Q - T^m) > v_r(T^m)$.*

Proof. We first check existence. Define sequences $\{P_l\}, \{Q_l\}$ as follows. Define $P_0 = 1$ and $Q_0 = T^m$. Given P_l and Q_l, write

$$R - P_l Q_l = \sum_i a_i T^i,$$

then put

$$X_l = \sum_{i \geq m} a_i T^{i-m}, \qquad Y_l = \sum_{i<m} a_i T^i$$

and set $P_{l+1} = P_l + X_l$, $Q_{l+1} = Q_l + Y_l$. Put $c_l = v_r(R - P_l Q_l) - rm$, so that $c_0 > 0$. Suppose that $v_r(P_l - 1) \geq c_0$, $v_r(Q_l - T^m) \geq c_0 + rm$, and $c_l \geq c_0$. Then visibly $v_r(P_{l+1} - 1) \geq c_0$ and $v_r(Q_{l+1} - T^m) \geq c_0 + rm$; by Lemma 3.1.5,

$$\begin{aligned} c_{l+1} &= v_r(R - (P_l + X_l)(Q_l + Y_l)) - rm \\ &= v_r(X_l(T^m - Q_l) + (1 - P_l)Y_l - X_l Y_l) - rm \\ &\geq \min\{c_l + (c_0 + rm), c_0 + (c_l + rm), c_l + (c_l + rm)\} - rm \\ &\geq c_l + c_0. \end{aligned}$$

By induction on l, we deduce that $c_l \geq (l+1)c_0$. Moreover, each P_l has degree at most $\deg(R) - m$, and each $Q_l - T^m$ has degree at most $m - 1$. Consequently, the sequences $\{P_l\}$ and $\{Q_l\}$ converge under v_r to polynomials P and Q, which have the desired properties.

We next check uniqueness. Suppose $R = P_1 Q_1$ is a second such factorization; put $c = \min\{v_r(P - P_1), v_r(Q - Q_1) - v_r(T^m)\}$. Put

$$X = R - P_1 Q = (P - P_1)Q = P_1(Q_1 - Q),$$

and suppose $X \neq 0$; then $v_r(X) = c + rm$ by Lemma 3.1.5. Write $X = \sum b_k T^k$, and choose k such that $v_r(X) = v_r(b_k T^k)$. The equality

$$X = (P - P_1)T^m + (P - P_1)(Q - T^m)$$

shows that we cannot have $k < m$, while the equality

$$X = Q_1 - Q + (P_1 - 1)(Q_1 - Q)$$

shows that we cannot have $k \geq m$. This contradiction forces $X = 0$, proving $P = P_1, Q = Q_1$ as desired. □

Remark 3.2.3. Note that the proof of Proposition 3.2.2 does not involve any divisions. Consequently, if the coefficients of P lie in a subring S of F which is complete under the norm, then the coefficients of Q and R also lie in S.

We obtain a corollary akin to a factorization result of Dwork-Robba [9, Theorem 6.2.3].

Corollary 3.2.4. *Any monic twisted polynomial $P \in F\{T\}$ admits a unique factorization*

$$P = P_{r_1} \cdots P_{r_m} P_+$$

for some $r_1 < \cdots < r_m < r_0$, where each P_{r_i} is monic with all slopes equal to r_i, and P_+ is monic with all slopes at least r_0.

Remark 3.2.5. Note that by Remark 3.1.3, Corollary 3.2.4 can also be stated with the factors in the reverse order; the degrees of the individual factors will not change, but the factors themselves may differ.

3.3 Differential modules

Remember that we are still assuming that F is complete (Hypothesis 3.2.1).

Definition 3.3.1. A *differential module* over F is a finite dimensional F-vector space V equipped with an action of ∂, or equivalently, a left $F\{T\}$-module which is finite over F. Given a basis B of V, we may equip V with the supremum norm with respect to B, and thus define operator and spectral norms $|\partial|_{V,B}$ and $|\partial|_{V,B,\mathrm{sp}}$. Changing B gives an equivalent norm on V, so the spectral norm $|\partial|_{V,B,\mathrm{sp}}$ does not depend on B; we thus write it as $|\partial|_{V,\mathrm{sp}}$.

Remark 3.3.2. We will also have occasion to speak about differential modules over fields equipped with multiple derivations, in which case the notation for the operator/spectral norm will indicate which derivation is being measured. See §3.4.

Remark 3.3.3. Instead of the spectral norm $|\partial|_{V,\mathrm{sp}}$, we will invariably consider the truncated spectral norm $\max\{|\partial|_{F,\mathrm{sp}}, |\partial|_{V,\mathrm{sp}}\}$. (It turns out that these coincide [19, Lemma 6.2.4], but we will not use that fact here.) The truncated spectral norm can be computed in terms of a basis of V as follows: if D_n denotes the matrix via which ∂^n acts on this basis, then

$$\max\{|\partial|_{F,\mathrm{sp}}, |\partial|_{V,\mathrm{sp}}\} = \max\{|\partial|_{F,\mathrm{sp}}, \limsup_{n\to\infty} |D_n|^{1/n}\}, \tag{3.3.3.1}$$

where the norm applied to D_n is the supremum over entries [4, Proposition 1.3].

Definition 3.3.4. Let V be a differential module over F. A *cyclic vector* for V is an element $\mathbf{v} \in V$ not contained in any proper differential submodule; it is equivalent to ask that $\mathbf{v}, \partial(\mathbf{v}), \dots, \partial^{n-1}(\mathbf{v})$ form a basis of V for $n = \dim_F(V)$. A cyclic vector defines an isomorphism $V \cong F\{T\}/F\{T\}P$ for some $P \in F\{T\}$.

Lemma 3.3.5. *Every differential module over F contains a cyclic vector.*

Proof. See, e.g., [8, Theorem III.4.2]. □

Lemma 3.3.6. *Let $P \in F\{T\}$ be a monic twisted polynomial and let $V = F\{T\}/F\{T\}P$ be the corresponding differential module. Then every short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ of differential modules arises uniquely from a factorization $P = P_1P_2$ of P into monic twisted polynomials, in which $V_1 \cong F\{T\}/F\{T\}P_1$ and $V_2 \cong F\{T\}/F\{T\}P_2$ and the map $V \to V_2$ is induced by the natural projection $F\{T\}/F\{T\}P \to F\{T\}/F\{T\}P_2$.*

Proof. The kernel of $F\{T\} \to V_2$ is a left ideal of $F\{T\}$, so it is generated by a unique monic P_2, giving the isomorphism $V_2 \cong F\{T\}/F\{T\}P_2$ and the factorization $P = P_1P_2$. We also have $V_1 \cong F\{T\}P_2/F\{T\}P$, and the latter is isomorphic to $F\{T\}/F\{T\}P_1$ via right division by P_2. □

The following is attributed "Dwork-Katz-Turritin" (sic) in [4, Théorème 1.5].

Theorem 3.3.7. *Let $P \in F\{T\}$ be a nonzero twisted polynomial with least slope r, and put $V = F\{T\}/F\{T\}P$. Then*

$$\max\{|\partial|_F, |\partial|_{V,\mathrm{sp}}\} = \max\{|\partial|_F, e^{-r}\}.$$

Proof. If P has a single slope r and that slope satisfies $r < r_0$, or if P has all slopes at least r_0, then we obtain the claim by using the basis $1, T, \dots, T^{\deg(P)-1}$ and invoking Lemma 3.1.7. Otherwise, we may apply Corollary 3.2.4 to reduce to such cases. □

Remark 3.3.8. The proof of [4, Théorème 1.5] contains a minor error in its implication $1 \Longrightarrow 2$: in its notation, one passes from K to an algebraic extension $K(z)$ without worrying about whether $\|D\|$ increases as a result. (In our notation, this amounts to passing from F to an extension without checking whether $|\partial|_F$ increases.) The proof of Theorem 3.3.7 shows that the final result is nonetheless correct, and indeed the proof is only slightly changed.

Lemma 3.3.9. *Let $P \in F\{T\}$ be a nonzero twisted polynomial with all slopes equal to $r < r_0$ (resp. all slopes at least r_0). Then every Jordan-Hölder factor W of $V = F\{T\}/F\{T\}P$ satisfies $|\partial|_{W,\mathrm{sp}} = e^{-r}$ (resp. $|\partial|_{W,\mathrm{sp}} \leq |\partial|_F$).*

Proof. We induct on $\dim_F(V)$. If V is irreducible, then Theorem 3.3.7 implies the claim. Otherwise, choose a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$; by Lemma 3.3.6, we have a factorization $P = P_1P_2$ such that $V_i \cong F\{T\}/F\{T\}P_i$ for $i = 1, 2$. By Corollary 3.1.6, P_1 and P_2 also have all slopes equal to r (resp. all slopes at least r_0), so we may apply the induction hypothesis to V_1, V_2 to conclude. □

Theorem 3.3.10. *Let V be a differential module over F. Then there is a unique decomposition*

$$V = V_+ \oplus \bigoplus_{r<r_0} V_r$$

of differential modules, such that each Jordan-Hölder factor W_+ of V_+ satisfies $|\partial|_{W_+,\mathrm{sp}} \leq |\partial|_F$, and each Jordan-Hölder factor W_r of V_r satisfies $|\partial|_{W_r,\mathrm{sp}} = e^{-r}$.

Proof. The decomposition is clearly unique if it exists. To produce it, we induct on $\dim_F(V)$. Choose a cyclic vector, let $V \cong F\{T\}/F\{T\}P$ be the resulting isomorphism, and let r_1 be the least slope of P. If $r_1 \geq r_0$, we may take $V = V_+$ and be done, so assume $r_1 < r_0$. If P has all slopes equal to r_1, then Lemma 3.3.9 implies that we may take $V = V_{r_1}$ and be done, so assume the contrary.

Apply Corollary 3.2.4 to factor $P = P_{r_1}Q$ with P_{r_1} having all slopes equal to r_1, and Q having all slopes greater than r_1. By Lemma 3.3.6, this factorization gives rise to an exact

sequence $0 \to V_1 \to V \to V_2 \to 0$ in which (by Lemma 3.3.9 and the induction hypothesis) each Jordan-Hölder factor of V_1 has spectral norm of ∂ equal to e^{-r_1}, and each Jordan-Hölder factor of V_2 has spectral norm of ∂ strictly less than e^{-r_1}.

Now apply Corollary 3.2.4 to factor P again, but this time in the opposite ring of $F\{T\}$ as per Remark 3.2.5. That is, write $P = Q'P'_{r_1}$ with P'_{r_1} having all slopes equal to r_1 and Q' having all slopes greater than r_1. Then Lemma 3.3.6 and Lemma 3.3.9 give an exact sequence $0 \to V'_1 \to V \to V'_2 \to 0$ in which each Jordan-Hölder factor of V'_2 has spectral norm e^{-r_1}, and each Jordan-Hölder factor of V'_1 has spectral norm strictly less than e^{-r_1}. In particular, $\dim(V_1) = \dim(V'_2)$, and $V_1 \cap V'_1 = \{0\}$; this forces $V \cong V_1 \oplus V'_1$. Splitting off $V_{r_1} = V_1$ and repeating, we obtain the desired decomposition. □

3.4 Differential fields of higher order

Hypothesis 3.4.1. We now modify Hypothesis 3.0.1 to say that F is a *complete valued differential field of order n* of characteristic zero. That is, in addition to being complete for a norm, F is equipped with not one but n commuting nonzero derivations $\partial_1, \dots, \partial_n$.

When comparing norms for different derivations acting on a differential module, it is useful to renormalize to remove the spectral norms of the derivations themselves.

Definition 3.4.2. A *differential module* over F is now a finite dimensional K-vector space V equipped with actions of $\partial_1, \dots, \partial_n$. Define the *scale* of V as

$$\max\left\{\max\left\{1, \frac{|\partial_i|_{V,\mathrm{sp}}}{|\partial_i|_{F,\mathrm{sp}}}\right\} : i = 1, \dots, n\right\}.$$

For each i at which the outer maximum is achieved, we say ∂_i is *dominant for V*.

Theorem 3.4.3. *Suppose that the $|\partial_i|_F/|\partial_i|_{F,\mathrm{sp}}$ for $i = 1, \dots, n$ are all equal to a common value s_0. Let V be a differential module over F. Then there is a unique decomposition*

$$V = V_+ \oplus \bigoplus_{s > s_0} V_s$$

such that each Jordan-Hölder factor of V_s has scale s, and each Jordan-Hölder factor of V_+ has scale at most s_0.

Proof. Apply Theorem 3.3.10 for each ∂_i; the uniqueness assertion in the proposition means that the decomposition with respect to ∂_i is respected by the other ∂_j. By taking the common refinement of these decompositions, then appropriately recombining terms, we obtain the desired result. □

Proposition 3.4.4. *Suppose that F is discretely valued, and the $|\partial_i|_F/|\partial_i|_{F,\mathrm{sp}}$ for $i = 1, \dots, n$ are all equal to a common value s_0. Let V be a differential module over F, and consider the decomposition in Theorem 3.4.3. Then for $s > s_0$, $s^{\dim(V_s)} \in s_0^{\mathbb{Z}}|F^*|$.*

Proof. Apply Theorem 3.3.10 to V_s for each ∂_i. From the result, we obtain a decomposition $V_s = \oplus V_{s,i}$ in which for each Jordan-Hölder factor W_i of $V_{s,i}$, we have that ∂_j is dominant for W_i when $j = i$ but not when $j = 1, \dots, i-1$.

Choose a cyclic vector for $V_{s,i}$ with respect to ∂_i; let $P(T) = T^d + \sum_{i=0}^{d-1} a_i T^i$ be the resulting twisted polynomial. By Corollary 3.2.4, the Newton polygon of P must have all slopes equal to $-\log(s|\partial_i|_{F,\mathrm{sp}})$; it follows that $(s|\partial_i|_{F,\mathrm{sp}})^d = |a_0|$.

Note that $|\partial_i|_F \in |F^*|$ because F is discretely valued, so $|\partial_i|_{F,\mathrm{sp}} = |\partial_i|_F / s_0 \in s_0^{\mathbb{Z}}|F^*|$, and so $s^{\dim(V_{s,i})} \in s_0^{\mathbb{Z}}|F^*|$. Since $\sum_i \dim(V_{s,i}) = \dim(V_s)$, $s^{\dim(V_s)} \in s_0^{\mathbb{Z}}|F^*|$, as desired. □

4 Generic radii of convergence

In this section, we revisit the usual notion of generic radii of convergence of differential equations from the work of Dwork, Robba, et al., but this time working in several dimensions.

4.1 Generalized polyannuli

It will be convenient to consider subsets of affine spaces more general than the polyannuli considered in [16, Definition 3.1.5].

Notation 4.1.1. For $X = (X_1, \dots, X_n)$ an n-tuple:

- for A an $n \times n$ matrix, write X^A for the n-tuple whose j-th entry is $\prod_{i=1}^n x_i^{A_{ij}}$;
- for B an n-tuple, put $X^B = X^A$ for A the diagonal matrix with $A_{ii} = B_i$;
- for c a number, put $X^c = X^A$ for A the scalar matrix cI_n.

Definition 4.1.2. By a *log-(rational polyhedral) subset*, or *log-RP subset*, of $(0, +\infty)^n$, we will mean a subset S whose image under the logarithm map to $\mathbb{R}^n$ is a rational polyhedral set in the sense of Definition 2.2.3. We say S is *ind-log-RP* if it is the union of an increasing sequence of log-RP subsets.

Notation 4.1.3. Let S be an ind-log-RP subset of $(0, +\infty)^n$. Write $A_K(S)$ for the rigid analytic subspace of $\mathbb{A}^n_K$ defined by the conditions

$$(|t_1|, \dots, |t_n|) \in S;$$

if S is log-RP and $\log(S)$ is bounded, then $A_K(S)$ is affinoid. Note that

$$\Gamma(A_K(S), \mathcal{O}) = \left\{ \sum_{J \in \mathbb{Z}^n} c_J T^J : c_J \in K, \lim_{J \to \infty} |c_J| R^J = 0 \quad (R \in S) \right\},$$

where $T = (t_1, \dots, t_n)$. (The limit condition should be interpreted as follows: for each $R \in S$ and each $\epsilon > 0$, there are only finitely many $J \in \mathbb{Z}^n$ with $|c_J| R^J > \epsilon$.) For $S = \{R\}$ a singleton set, we write $A_K(R)$ for $A_K(S)$.

The following toric coordinate changes will be useful.

Definition 4.1.4. For A an $n \times n$ matrix, let $f_A : \mathbb{G}_m^n \to \mathbb{G}_m^n$ be the map $T \mapsto T^A$, or the induced map $A_K(R) \to A_K(R^A)$.

Lemma 4.1.5. *For any complete extension K' of K, any $c_1, \dots, c_n \in K'$ with $|c_i| = r_i$, and any $\lambda \in (0,1]$, define the open polydisc*

$$D(C, \lambda R) = \{T \in A_K(R) : |t_i - c_i| < \lambda r_i \quad (i = 1, \dots, n)\}.$$

Then f_A carries $D(C, \lambda R)$ to $D(C^A, \lambda R^A)$.

Proof. Rewrite the defining condition of $D(C, \lambda R)$ as $|1 - t_i/c_i| < \lambda$. Then note that this implies $|1 - (t_i/c_i)^n| < \lambda$ for any $n \in \mathbb{Z}$, by examination of the binomial expansion of $(1 - (1 - t_i/c_i))^n$. To finish, recall that for $\lambda \in (0,1]$, $|1 - a|, |1 - b| < \lambda$ implies $|1 - ab| < \lambda$ because $1 - ab = (1 - a) + (1 - b) - (1 - a)(1 - b)$. $\square$

Definition 4.1.6. For $R = (r_1, \dots, r_n) \in S$, the space $A_K(S)$ carries a Gauss norm $|\cdot|_R$ defined by

$$\left| \sum_J c_J T^J \right|_R = \sup_J \{|c_J| R^J\};$$

it is in fact the supremum norm on $A_K(R)$.

The following convexity lemma (analogous to the Hadamard three circles theorem) is a repackaging of [16, Lemma 3.1.6], but similar observations occur much earlier in the literature, e.g., [1, Corollaire 4.2.8], [6, Corollaire 5.4.9].

Lemma 4.1.7. *For $A, B \in S$ and $c \in [0,1]$, put $R = A^c B^{1-c}$; that is, $r_i = a_i^c b_i^{1-c}$ for $i = 1, \dots, n$. Then for any $f \in \Gamma(A_K(S), \mathcal{O})$,*

$$|f|_R \le |f|_A^c |f|_B^{1-c}.$$

Proof. Since each Gauss norm is calculated as a supremum over monomials, it suffices to check the inequality in the case of a single monomial, in which case it becomes an equality. $\square$

4.2 Generic radii of convergence

Definition 4.2.1. Let S be a log-RP subset of $(0, +\infty)^n$, take $R \in S$, and let $\mathcal{E}$ be a ∇-module (locally free coherent sheaf plus integrable connection) on $A_K(S)$. Let F (or F_R in case of ambiguity) be the completion of $\operatorname{Frac} \Gamma(A_K(S), \mathcal{O})$ under $|\cdot|_R$, and put

$$V = \Gamma(A_K(S), \mathcal{E}) \otimes_{\Gamma(A_K(S), \mathcal{O})} F.$$

For $i = 1, \dots, n$, define $\partial_i = \frac{\partial}{\partial t_i}$ as a derivation on F. View F as a differential field of order n, view V as a differential module over F, and let $T(\mathcal{E}, R)$ be the reciprocal of the scale of V; that is,

$$T(\mathcal{E}, R) = \min_i \{\min\{1, |\partial_i|_{V,\mathrm{sp}}^{-1} |\partial_i|_{F,\mathrm{sp}}\}\}.$$

Remark 4.2.2. We may interpret $T(\mathcal{E}, R)$ as the largest $\lambda \in (0,1]$ such that for any complete extension K' of K and any $C = (c_1, \dots, c_n) \in (K')^n$ with $|c_i| = r_i$ for $i = 1, \dots, n$, $\mathcal{E}$ admits a basis of horizontal sections on $D(C, \lambda R)$. In particular, for $n = 1$, our function $T(\mathcal{E}, R)$ is equal to R^{-1} times the generic radius of convergence $R(\mathcal{E}, R)$ of [4]. The letter T is used here to denote "toric normalization".

Remark 4.2.3. It may be helpful to compare Remark 4.2.2 with [12, Definition 5.3], but one must beware of three typos in the latter: the min should be a max, the subscript ρ is missing, and the reference to [4, Proposition 1.2] should be to Proposition 1.3 therein.

Remark 4.2.4. The following are easily verified.

- If $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ is exact, then
$$T(\mathcal{E}, R) = \min\{T(\mathcal{E}_1, R), T(\mathcal{E}_2, R)\}.$$
- We have
$$T(\mathcal{E}_1 \otimes \mathcal{E}_2, R) \leq \min\{T(\mathcal{E}_1, R), T(\mathcal{E}_2, R)\}.$$
- We have
$$T(\mathcal{E}^\vee, R) = T(\mathcal{E}, R).$$

The function T also satisfies a toric invariance property.

Proposition 4.2.5. *Let S be a log-RP subset of $(0, +\infty)^n$, take $R \in S$, and let $\mathcal{E}$ be a ∇-module on $A_K(S)$. For $A \in M_n(\mathbb{Z})$, put $S^A = \{R^A : R \in S\}$. Then for any ∇-module $\mathcal{E}$ on $A_K(S^A)$,*
$$T(f_A^* \mathcal{E}, R) \geq T(\mathcal{E}, R^A),$$
with equality if $A \in \mathrm{GL}_n(\mathbb{Z})$.

Proof. This follows immediately from Lemma 4.1.5. □

Lemma 4.1.7 yields the following log-concavity property, which generalizes [4, Proposition 2.3].

Proposition 4.2.6. *Let S be a log-RP subset of $(0, +\infty)^n$. Let $\mathcal{E}$ be a ∇-module on $A_K(S)$. For $A, B \in S$ and $c \in [0,1]$,*
$$T(\mathcal{E}, A^c B^{1-c}) \geq T(\mathcal{E}, A)^c T(\mathcal{E}, B)^{1-c}.$$

Proof. Since S is log-RP, $A_K(S)$ is affinoid; by Kiehl's theorem, $\mathcal{E}$ is generated by finitely many global sections. Let $\mathbf{e}_1, \dots, \mathbf{e}_m$ be a maximal linearly independent set of global sections, and let $D_{i,l}$ be the matrix over $\operatorname{Frac} \Gamma(A_K(S), \mathcal{O})$ via which $\frac{\partial^l}{\partial t_i^l}$ acts on $\mathbf{e}_1, \dots, \mathbf{e}_m$. Since $\mathbf{e}_1, \dots, \mathbf{e}_m$ are maximal linearly independent, we can choose $f \in \Gamma(A_K(S), \mathcal{O})$ so that $f\Gamma(A_K(S), \mathcal{E})$ is contained in the span of $\mathbf{e}_1, \dots, \mathbf{e}_m$. This implies that $f D_{i,l}$ has entries in $\Gamma(A_K(S), \mathcal{O})$ for all i, l.

Put $R = A^c B^{1-c}$. By Lemma 4.1.7, we have

$$|f|_R |D_{i,l}|_R \leq |f|_A^c |D_{i,l}|_A^c |f|_B^{1-c} |D_{i,l}|_B^{1-c};$$

taking l-th roots of both sides and taking limits superior yields

$$\max\{|\partial_i|_{F_R,\mathrm{sp}}, \limsup_{l\to\infty} |D_{i,l}|_R^{1/l}\} \leq \max\left\{|\partial_i|_{F_A,\mathrm{sp}}, \left(\limsup_{l\to\infty} |D_{i,l}|_A^{1/l}\right)^c\right\} \cdot \max\left\{|\partial_i|_{F_B,\mathrm{sp}}, \left(\limsup_{l\to\infty} |D_{i,l}|_B^{1/l}\right)^{1-c}\right\}$$

because the factors coming from f all tend to 1. By (3.3.3.1), this yields the desired result. $\square$

Example 4.2.7. Let $\mathcal{E}$ be the ∇-module of rank 1 defined by $\nabla \mathbf{v} = \lambda \pi d(t_1^{i_1} \cdots t_n^{i_n})$, where $\lambda \in \mathfrak{o}_K^*$, $\pi \in K$ satisfies $\pi^{p-1} = -p$ (that is, π is a *Dwork pi* and $\mathcal{E}$ is a *Dwork isocrystal*), and $i_1, \dots, i_n \in \mathbb{Z}$ are not all divisible by p. Then as in [12, Chapter 5], one may check that

$$T(\mathcal{E}, R) = \min\{1, r_1^{-i_1} \cdots r_n^{-i_n}\}.$$

4.3 The Frobenius antecedent theorem

We now revisit the Frobenius antecedent theorem of Christol-Dwork [4, Théorème 5.4] in a higher-dimensional context, following [12, Theorem 6.15].

Hypothesis 4.3.1. Let Y be an affinoid space over K, and suppose $t_1, \dots, t_n \in \Gamma(Y, \mathcal{O})^*$ are such that $dt_1, \dots, dt_n$ freely generate $\Omega^1_{Y/K}$; let $f : Y \to \mathbb{A}^n_K$ be the resulting étale morphism. Form the Cartesian diagram

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d] & Y \arrow[d, "f"] \\ \mathbb{A}^n_K \arrow[r] & \mathbb{A}^n_K \end{tikzcd} \qquad (4.3.1.1)$$

in which the morphism $\mathbb{A}^n_K \to \mathbb{A}^n_K$ is given by $t_i \mapsto t_i^p$ $(i = 1, \dots, n)$. Let $\mathcal{E}'$ be a ∇-module on Y' such that

$$\left|\frac{\partial}{\partial t_i}\right|_{\mathcal{E}',\mathrm{sp}} < |t_i|^{-1}_{\sup,Y} \qquad (i = 1, \dots, n), \qquad (4.3.1.2)$$

where the left-hand side is computed using any norm on $\Gamma(Y', \mathcal{E}')$ compatible with the affinoid norm on $\Gamma(Y', \mathcal{O})$. (Since any two such norms are equivalent, the spectral norm does not depend on the choice.)

Definition 4.3.2. Suppose that K contains a primitive p-th root of unity ζ. For $J = (j_1, \dots, j_n) \in (\mathbb{Z}/p\mathbb{Z})^n$, let $g_J : Y' \to Y'$ be the map defined by $t_i \mapsto t_i \zeta^{j_i}$ for $i = 1, \dots, n$.

(More precisely, we get g_J from the Cartesian square (4.3.1.1) using the original map $Y' \to Y$ and the map $Y' \to \mathbb{A}^n_K$ given by $t_1\zeta^{j_1}, \dots, t_n\zeta^{j_n}$.) Then the map $h_J : g_J^*\mathcal{E}' \to \mathcal{E}'$ defined by

$$h_J(\mathbf{v}) = \sum_{i_1,\dots,i_n=0}^{\infty} (\zeta^{j_1}-1)^{i_1}\cdots(\zeta^{j_n}-1)^{i_n} \frac{t_1^{i_1}\cdots t_n^{i_n}}{i_1!\cdots i_n!} \frac{\partial^{i_1}}{\partial t_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial t_n^{i_n}} \mathbf{v}$$

converges because of (4.3.1.2).

Proposition 4.3.3. *Suppose that K contains a primitive p-th root of unity ζ. Under Hypothesis 4.3.1, there is a unique ∇-module $\mathcal{E}$ on Y such that $g^*\mathcal{E} \cong \mathcal{E}'$ and the action of the h_J on $\mathcal{E}'$ is induced by the trivial action on $\mathcal{E}$.*

Proof. Put $M' = \Gamma(Y', \mathcal{E}')$. The maps h_J satisfy $h_J(t_i\mathbf{v}) = \zeta^{j_i} t_i h_J(\mathbf{v})$; hence for $J = (j_1, \dots, j_n) \in \{0, \dots, p-1\}^n$, if we define

$$f_J(\mathbf{v}) = t_1^{-j_1} \cdots t_n^{-j_n} \sum_{J' \in (\mathbb{Z}/p\mathbb{Z})^n} \zeta^{-j_1 j'_1 - \cdots - j_n j'_n} h_{J'}(\mathbf{v}),$$

then $f_J(\mathbf{v})$ is fixed by the $h_{J'}$. Let M be the $\Gamma(Y', g^{-1}(\mathcal{O}))$-span of the $f_J(\mathbf{v})$; then M is a coherent $\Gamma(Y, \mathcal{O})$-module, and (by an appropriate form of Hilbert's Theorem 90) the natural map $M \otimes \Gamma(Y', \mathcal{O}) \to M'$ is a $(\mathbb{Z}/p\mathbb{Z})^n$-equivariant isomorphism. We give M a ∇-module structure by declaring the action of $\frac{\partial}{\partial t_i}$ on M to be $p^{-1} t_i^{1-p}$ times the action of $\frac{\partial}{\partial t_i}$ on M'. This gives rise to $\mathcal{E}$ such that $\mathcal{E}' \cong g^*\mathcal{E}$, which evidently is unique for the property of being fixed by the h_J. □

Definition 4.3.4. Under Hypothesis 4.3.1, we call $\mathcal{E}'$ the *Frobenius antecedent* of $\mathcal{E}$. Note that the uniqueness implies that it makes sense to define a Frobenius antecedent for a ∇-module on a rigid space Y even if (4.3.1.2) is only satisfied after replacing Y with each element of an admissible open cover, or if K does not contain a primitive p-th root of unity.

4.4 Frobenius antecedents and generic radii

Notation 4.4.1. Throughout this subsection, write $S^{1/p} = \{R^{1/p} : R \in S\}$ for $S \subseteq (0, +\infty)^n$, and let f_p denote the map $f_{pI_n} : A_K(S^{1/p}) \to A_K(S)$ for any S.

Lemma 4.4.2. *Let S be a log-RP subset of $(0, +\infty)^n$, suppose $(1, \dots, 1, \rho) \in S$, and let $\mathcal{E}$ be a ∇-module on $A_K(S)$. Then*

$$T(f_p^*\mathcal{E}, (1, \dots, 1, \rho^{1/p})) \geq T(\mathcal{E}, (1, \dots, 1, \rho))^{1/p}.$$

This inequality can be shown to be an equality when $T(\mathcal{E}, (1, \dots, 1, \rho)) > |p|^{p/(p-1)}$, but we will not need that more precise result here.

Proof. It suffices to observe that for $R = (1, \dots, 1, \rho)$, f_p carries $D(C^{1/p}, \lambda^{1/p} R^{1/p})$ into $D(C, \lambda R)$. The latter follows from [12, Lemma 5.12], but note a misprint therein: in the last line of the statement, the quantities $r\rho^{1/p}$ and $r^p\rho$ should be $r^{1/p}\rho^{1/p}$ and $r\rho$, respectively. □

Theorem 4.4.3. *Put $S = [1,1]^{n-1} \times (\epsilon, 1)$ for some $\epsilon \in (0,1)$. Let $\mathcal{F}$ be a ∇-module on $A_K(S^{1/p})$ such that*

$$T(\mathcal{F}, (1, \dots, 1, \rho^{1/p})) > |p|^{1/(p-1)} \qquad (\rho \in (\epsilon, 1)). \tag{4.4.3.1}$$

Then $\mathcal{F}$ admits a Frobenius antecedent $\mathcal{E}$ on $A_K(S)$, which satisfies

$$T(\mathcal{E}, (1, \dots, 1, \rho))^{1/p} = T(\mathcal{F}, (1, \dots, 1, \rho^{1/p})) \qquad (\rho \in (\epsilon, 1)). \tag{4.4.3.2}$$

Proof. For each point in S, (4.4.3.1) implies that one can find a neighborhood S' of that point in S such that on $A_K(S')$, (4.3.1.2) holds. We then glue to obtain a Frobenius antecedent on all of $A_K(S)$.

To prove (4.4.3.2), note that with $R = (1, \dots, 1, \rho)$, given $c_1, \dots, c_n$ with $|c_i| = 1$ for $i = 1, \dots, n-1$ and $|c_n| = \rho^{1/p} \in (\epsilon^{1/p}, 1)$, we can apply the maps f_J (from the proof of Proposition 4.3.3) to horizontal sections on a polydisc $D(C, \lambda^{1/p} R^{1/p})$ to obtain horizontal sections on $D(C^p, \lambda R)$. Consequently,

$$T(\mathcal{E}, (1, \dots, 1, \rho))^{1/p} \geq T(\mathcal{F}, (1, \dots, 1, \rho^{1/p})) \qquad (\rho \in (\epsilon, 1));$$

the reverse inequality follows from Lemma 4.4.2. □

Using Frobenius antecedents, one overcomes the scale barrier built into the results of Section 3.

Lemma 4.4.4. *Take S as in Theorem 4.4.3, and let $\mathcal{E}$ be a ∇-module on $A_K(S)$. Then for each $\rho \in (\epsilon, 1)$, there exists an integer $j \in \{1, \dots, \operatorname{rank}(\mathcal{E})\}$ and a nonnegative integer m such that*

$$T(\mathcal{E}, (1, \dots, 1, \rho))^j \in \rho^{\mathbb{Z}} (|K^*| |p|^{(1/(p-1))\mathbb{Z}})^{p^{-m}}.$$

Proof. Let m be the least nonnegative integer such that

$$T(\mathcal{E}, (1, \dots, 1, \rho))^{p^m} \leq |p|^{1/(p-1)}.$$

If $T(\mathcal{E}, (1, \dots, 1, \rho))^{p^m} = |p|^{1/(p-1)}$, then we are done, so assume not. By Proposition 4.2.6, $T(\mathcal{E}, (1, \dots, 1, \rho))$ is a log-concave and hence continuous function of ρ, so we can choose a closed interval I with endpoints in the divisible closure of $|K^*|$, such that $T(\mathcal{E}, (1, \dots, 1, \eta)) > |p|^{p^{1-m}/(p-1)}$ for $\eta \in I$. Apply Theorem 4.4.3 m times to produce a ∇-module $\mathcal{E}'$ with

$$T(\mathcal{E}, (1, \dots, 1, \rho)) = T(\mathcal{E}', (1, \dots, 1, \rho^{p^m}))^{1/p^m}.$$

Then apply Proposition 3.4.4 to $\mathcal{E}'$, noting that for the derivation $\frac{\partial}{\partial t_i}$ on $\operatorname{Frac} \Gamma(A_K(S), \mathcal{O})$ under the R-Gauss norm, the operator norm and spectral norm are r_i^{-1} and $|p|^{1/(p-1)} r_i^{-1}$, respectively. This yields the desired result. (Compare [5, Théorème 4.2-1].) □

Lemma 4.4.5. *Take S as in Theorem 4.4.3, and let $\mathcal{E}$ be a ∇-module on $A_K(S)$. Suppose that $T(\mathcal{E}, (1, \dots, 1, \rho)) \to 1$ as $\rho \to 1^-$. Then there exist $\eta \in [\epsilon, 1)$, an integer $1 \leq j \leq \operatorname{rank}(\mathcal{E})$, and a nonnegative integer i such that $T(\mathcal{E}, (1, \dots, 1, \rho)) = \rho^{i/j}$ for $\rho \in (\eta, 1)$.*

Proof. There is no harm in assuming that $|p|^{1/(p-1)} \in |K^*|$. For $c \in (0, -\log(\epsilon))$, define

$$f(c) = \log T(\mathcal{E}, (1, \dots, 1, e^{-c}));$$

this function is concave by Proposition 4.2.6, takes nonpositive values, and by hypothesis has limit 0 as $c \to 0^+$. Consequently, f is nonincreasing.

For i a sufficiently large integer, we can find $c_i \in (0, -\log(\epsilon))$ such that

$$f(c_i) = \frac{1}{p^m(p-1)} \log |p|;$$

the c_i then form a decreasing sequence. By Lemma 4.4.4, for each $c \in (c_{i+1}, c_i) \cap \mathbb{Q} \log |p|$, there exists $j \in \{1, \dots, \mathrm{rank}(\mathcal{E})\}$ such that

$$f(c) \in \frac{1}{j}(p^{-m-1} \log |K^*| + \mathbb{Z}c).$$

By Theorem 2.3.2, f is piecewise affine on (c_{i+1}, c_i), and each slope is a rational number with denominator bounded by $\mathrm{rank}(\mathcal{E})$. In particular, the slopes of f belong to a discrete subgroup of $\mathbb{R}$.

As $c \to 0^+$, the slopes of f on successive domains of affinity form a nondecreasing sequence of values, each of which is nonpositive because f is nonincreasing. Since these values lie in a discrete subgroup of $\mathbb{R}$, they must stabilize; that is, f is affine in some neighborhood of 0. Since $f \to 0$ as $c \to 0^+$, f must actually be linear in a neighborhood of 0. This yields the desired result. (Compare [5, Théorème 4.2-1].) $\square$

Definition 4.4.6. We say an n-tuple $R \in (0, +\infty)^n$ is *commensurable* if $r_1, \dots, r_n$ generate a discrete subgroup of the multiplicative group $\mathbb{R}_{>0}$. In this case, we call the generator of that subgroup lying in $(0, 1)$ the *generator* of R.

Theorem 4.4.7. *Let $R \neq (1, \dots, 1) \in (0, +\infty)^n$ be commensurable with generator ρ. Let S be an ind-log-RP subset of $(0, +\infty)^n$ containing R^c for all $c > 0$ sufficiently small. Let $\mathcal{E}$ be a ∇-module on $A_K(S)$ such that $T(\mathcal{E}, R^c) \to 1$ as $c \to 0^+$. Then there exist integers i, j with $i \geq 0$ and $1 \leq j \leq \mathrm{rank}(\mathcal{E})$, such that*

$$T(\mathcal{E}, R^c) = \rho^{i/j} \qquad \text{for } c > 0 \text{ sufficiently small.}$$

Proof. This reduces to Lemma 4.4.5 by applying a suitable toric change of coordinates f_A. $\square$

Remark 4.4.8. As in the one-dimensional case [5, Proposition 6.3-11], one can enforce the condition that $T(\mathcal{E}, R^c) \to 1$ as $c \to 0^+$ by equipping $\mathcal{E}$ with a Frobenius structure. Explicitly, suppose that q is a power of p, and that $\sigma : A_K(S^{1/q}) \to A_K(S)$ is a map obtained by composing the toric map f_{qI_n} with a q-power Frobenius lift on K. If there is an isomorphism $\sigma^*\mathcal{E} \cong \mathcal{E}$ over $A_K(S^{1/q})$, then Lemma 4.4.2 implies that for $R \in S$, $T(\mathcal{E}, R^{1/q^m}) \geq T(\mathcal{E}, R)^{1/q^m}$, so the values of $T(\mathcal{E}, R^c)$ get arbitrarily close to 1; by Proposition 4.2.6, it follows that $T(\mathcal{E}, R^c) \to 1$ as $c \to 0^+$.

5 Around the local monodromy theorem

In this section, we recall the p-adic local monodromy theorem, in a generalized form suited to treating monomial valuations. We then mention some related results, on the interplay between generic radii of convergence in the one-dimensional case and local monodromy.

5.1 The monodromy theorem for fake annuli

To state the monodromy theorem at the level of generality we need, we must recall some terminology from [14].

Definition 5.1.1. We say a linear functional $\lambda : \mathbb{R}^n \to \mathbb{R}$ is *irrational* if $\mathbb{Z}^n \cap \ker(\lambda) = \{0\}$. For λ an irrational functional, write $\lambda_1, \dots, \lambda_n$ for the images under λ of the standard generators of $\mathbb{Z}^n$. For $I \subseteq (0,1)$, let $\mathcal{R}_I^\lambda$ (resp. $\mathcal{R}_I^{\lambda,\mathrm{int}}$) be the Fréchet completion of $K[t_1^\pm, \dots, t_n^\pm]$ (resp. $\mathfrak{o}_K[t_1^\pm, \dots, t_n^\pm]$) with respect to the Gauss norms $|\cdot|_{\rho^{\lambda_1}, \dots, \rho^{\lambda_n}}$ for $\rho \in I$. Write $\mathcal{R}^\lambda$ (resp. $\mathcal{R}^{\lambda,\mathrm{int}}$) for the union of $\mathcal{R}_{[\rho,1)}^\lambda$ (resp. $\mathcal{R}_{[\rho,1)}^{\lambda,\mathrm{int}}$) over all $\rho \in (0,1)$.

Remark 5.1.2. In our notation for generalized polyannuli, $\mathcal{R}_I^\lambda$ would be the global sections of the structure sheaf on $A_K(S)$ for

$$S = \{(\rho^{\lambda_1}, \dots, \rho^{\lambda_n}) : \rho \in I\}$$

if the latter were an ind-log-RP subset; however, that can only happen when I consists of a single point, or when $n = 1$ (the case of a true annulus). This is what is fake about a so-called fake annulus: it does not fit any conventional definition of an analytic subspace of $\mathbb{A}_K^n$, even in Berkovich's framework for nonarchimedean analytic geometry [3].

Remark 5.1.3. Given an interval I, let I' be the interval consisting of those $r \in (0, +\infty)$ such that $|p|^{r/w(p)} \in I$. For λ an irrational functional, the ring $\Gamma_{I'}^\lambda$ of [14, Definition 2.4.1] (with the lattice therein taken to be $\mathbb{Z}^n$) is isomorphic to $\mathcal{R}_I^\lambda$ via a map sending $\{z_i\}$ to t_i for $i = 1, \dots, n$. This identification has a number of consequences, some captured in Lemma 5.1.4 below.

Lemma 5.1.4. *(a) For I closed, $\mathcal{R}_I^\lambda$ is a principal ideal domain.*

(b) For any $\rho \in (0,1)$, $\mathcal{R}_{[\rho,1)}^\lambda$ is a Bézout domain (an integral domain whose finitely generated ideals are principal).

(c) Let $I_1 \subset I_2 \subset \cdots$ be an increasing sequence of closed intervals with union $[\rho, 1)$. Given any sequence $M_1, M_2, \dots$ in which M_l is a finite free $\mathcal{R}_{I_l}^\lambda$-module, together with isomorphisms $\iota_l : M_{l+1} \otimes \mathcal{R}_{I_l}^\lambda \cong M_l$, there exist a finite free $\mathcal{R}_{[\rho,1)}^\lambda$-module M and isomorphisms $\psi_l : M \otimes \mathcal{R}_{I_l}^\lambda \cong M_l$ such that $\iota_l \circ \psi_{l+1} = \psi_l$; moreover, M and the ψ_l are determined up to unique isomorphism.

Proof. For (a), see [13, Proposition 2.6.8]. For (b), see [13, Theorem 2.9.6]. For (c), see [13, Theorem 2.8.4]. □

Definition 5.1.5. Define a ∇-*module* over $\mathcal{R}^\lambda$ as a finite free $\mathcal{R}^\lambda$-module M equipped with an integrable connection $\nabla : M \to M \otimes \Omega^1_{\mathcal{R}^\lambda/K}$. We say a ∇-module over $\mathcal{R}^\lambda$ is *constant* if it has a basis of horizontal sections, *quasi-constant* if it becomes constant after tensoring with a finite étale extension of $\mathcal{R}^{\lambda,\mathrm{int}}$, and *(quasi)-unipotent* if it admits a filtration by ∇-submodules whose successive quotients are (quasi)-constant.

Definition 5.1.6. Let $\sigma : \mathcal{R}^\lambda \to \mathcal{R}^\lambda$ be a continuous endomorphism lifting a power of the absolute Frobenius map on the residue field of $\mathcal{R}^{\lambda,\mathrm{int}}$. Define an F-*module* (resp. (F,∇)-module) over $\mathcal{R}^\lambda$ relative to σ as a finite free $\mathcal{R}^\lambda$-module (resp. ∇-module) M equipped with an isomorphism $F : \sigma^* M \to M$ of modules (resp. of ∇-modules). As with true annuli, the category of (F,∇)-modules over $\mathcal{R}^\lambda$ is canonically independent of the choice of σ [14, Proposition 3.4.7].

Definition 5.1.7. For $s = c/d \in \mathbb{Q}$, an F-module M is *pure* (or *isoclinic*) *of slope* s if there exists a basis of M on which F^d acts via the product of a scalar of valuation c with an invertible matrix over $\mathcal{R}^{\lambda,\mathrm{int}}$. Note that this is the equivalent characterization of [13, Proposition 6.3.5] rather than the original definition; one can in fact develop the slope theory for F-modules using this definition instead, as in [18].

In this language, one has the following result from [14].

Theorem 5.1.8. *Let $\mathcal{E}$ be an (F,∇)-module over $\mathcal{R}^\lambda$.*

(a) There exists a unique filtration $0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m \subset \mathcal{E}$ of $\mathcal{E}$ by (F,∇)-submodules such that each $\mathcal{E}_i/\mathcal{E}_{i-1}$ is pure of some slope s_i as an F-module, and $s_1 < \cdots < s_m$.

(b) Each successive quotient of the filtration in (a) is quasi-constant as a ∇-module. Consequently, $\mathcal{E}$ is quasi-unipotent as a ∇-module.

Proof. Statement (a) is [14, Theorem 5.2.1]; note that this depends on the generalized slope filtration theorem of [13], not just on the original form of the theorem of [11]. Statement (b) is [14, Theorem 5.2.4]. ◻

5.2 Monodromy and convergence (one-dimensional case)

We now revert from fake annuli back to true annuli, to recall some results relating generic radii of convergence to wild ramification. We defer to [12] for a more extensive discussion of the points we only summarize here, including attributions.

Notation 5.2.1. Throughout this subsection, we take $n = 1$, drop λ, and write t for t_1. Also, as in [16], when we write an interval I out explicitly, we typically omit the parentheses in the notation $A_K(I)$.

Proposition 5.2.2. *The category of quasi-unipotent ∇-modules over $\mathcal{R}$ is equivalent to the category of representations of*

$$\mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t))) \times K$$

in finite dimensional K^{unr}-vector spaces, which are semilinear and permissible (the restriction to some open subgroup is trivial) on the first factor, and algebraic, K-rational, and unipotent on the second factor.

Proof. See [12, Theorem 4.45]. $\square$

Definition 5.2.3. Let $\mathcal{E}$ be a ∇-module on $A_K(\epsilon, 1)$ for some $\epsilon \in (0, 1)$. By Lemma 5.1.4(c), $\mathcal{E}$ corresponds to a ∇-module over $\mathcal{R}_{(\epsilon,1)}$; let $M_{\mathcal{E}}$ be the corresponding ∇-module over $\mathcal{R}$.

Proposition 5.2.4. *Assume that the field k is perfect. Let $\mathcal{E}$ be a ∇-module on $A_K(\epsilon, 1)$ for some $\epsilon \in (0, 1)$ such that $M_{\mathcal{E}}$ is quasi-unipotent. Then for $\rho \in (0, 1)$ sufficiently close to 1, $T(\mathcal{E}, \rho) = \rho^{\beta}$ for β equal to the highest ramification break of the Galois factor of the representation associated to $M_{\mathcal{E}}$ by Proposition 5.2.2. Moreover, if $\beta > 0$ and the lowest ramification break is also equal to β, then for $\rho \in (0, 1)$ sufficiently close to* 1, every *nonzero local horizontal section of $\mathcal{E}$ around a generic point of radius ρ has exact radius of convergence $\rho^{\beta+1}$.*

Proof. See [12, Theorem 5.23]. $\square$

Corollary 5.2.5. *Let $\mathcal{E}$ be a ∇-module on $A_K(\epsilon, 1)$ for some $\epsilon \in (0, 1)$, such that $M_{\mathcal{E}}$ is quasi-unipotent. Then the following are equivalent.*

(a) There exists a positive integer m coprime to p such that $M_{\mathcal{E}} \otimes \mathcal{R}[t^{1/m}]$ is unipotent.

(b) $T(\mathcal{E}, \rho) = 1$ for $\rho \in (\epsilon, 1)$ sufficiently close to 1.

(c) $T(\mathcal{E}, \rho) > \rho^{1/\operatorname{rank}(\mathcal{E})}$ for $\rho \in (\epsilon, 1)$ sufficiently close to 1.

Proof. There is no harm in enlarging K, so we may assume k is perfect. Clearly (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c). Given (c), by Proposition 5.2.4, the highest ramification break of the corresponding Galois representation must be less than $1/\operatorname{rank}(\mathcal{E})$; since the highest break must be a nonnegative rational number with denominator at most $\operatorname{rank}(\mathcal{E})$ (by the Hasse-Arf theorem), it must equal 0, that is, the representation is only tamely ramified. This yields the claim. $\square$

5.3 Monodromy and convergence (relative case)

In light of Proposition 5.2.4, it is natural to make the following definition.

Definition 5.3.1. With notation as in Theorem 4.4.7, we call the rational number i/j the *(differential) highest ramification break* of $\mathcal{E}$ in the direction of R, denoted $b(\mathcal{E}, R)$.

Proposition 5.3.2. *Let $A, B \in (0, +\infty)^n$ be commensurable, take $c \in [0, 1] \cap \mathbb{Q}$, put $R = A^c B^{1-c}$, and suppose R is also commensurable. Let α, β, ρ be the generators of A, B, R, respectively. Let S be a ind-log-RP subset of $(0, +\infty)^n$ which contains A^h, B^h, R^h for $h > 0$ sufficiently small. Let $\mathcal{E}$ be a ∇-module on $A_K(S)$ such that $T(\mathcal{E}, *^h) \to 1$ as $h \to 0^+$ for $* \in \{A, B, R\}$. Then*

$$\rho^{b(\mathcal{E},R)} \geq \alpha^{cb(\mathcal{E},A)} \beta^{(1-c)b(\mathcal{E},B)}.$$

Proof. Apply Proposition 4.2.6. ☐

Definition 5.3.3. Take S as in Theorem 4.4.3, and let $\mathcal{E}$ be a ∇-module on $A_K(S)$. Let L be the completion of $K(t_1, \dots, t_{n-1})$ under the $(1, \dots, 1)$-Gauss norm. Let $\mathcal{E}$ be a ∇-module on $A_K(S)$. Let $I_1 \subset I_2 \subset \cdots$ be an increasing sequence of closed intervals with union $(\epsilon, 1)$. Put $S_l = [1,1]^{n-1} \times I_l$, and put

$$M_l = \Gamma(A_K(S_l), \mathcal{E}) \otimes_{\Gamma(A_K(S_l), \mathcal{O})} \Gamma(A_L(I_l), \mathcal{O});$$

then there is a unique locally free coherent sheaf $\mathcal{F}$ on $A_L(\epsilon, 1)$ admitting identifications $M_l \cong \Gamma(A_L(I_l), \mathcal{F})$ compatible with restriction. Moreover, $\mathcal{F}$ inherits the structure of a ∇-module relative to L. We call $\mathcal{F}$ the *generic fibre* of $\mathcal{E}$; note that

$$T(\mathcal{E}, (1, \dots, 1, \rho)) \leq T(\mathcal{F}, \rho) \qquad (\rho \in (\epsilon, 1)). \tag{5.3.3.1}$$

Proposition 5.3.4. *Take S as in Theorem 4.4.3. Let $\mathcal{E}$ be a ∇-module on $A_K(S)$ such that $T(\mathcal{E}, (1, \dots, 1, \rho)) = 1$ for all $\rho \in (\epsilon, 1)$, and suppose that the generic fibre of $\mathcal{E}$ is quasi-unipotent. Then there exists a positive integer m coprime to p such that $f^*_{mI_n}\mathcal{E}$ is unipotent on $A_K([1,1]^{n-1}) \times A_K(\eta, 1)$, in the sense of [16, §3.2].*

Proof. By (5.3.3.1) and Corollary 5.2.5, we can choose m so that the generic fibre of $f^*_{mI_n}\mathcal{E}$ is unipotent. The claim then follows from [16, Proposition 3.4.3]. ☐

Remark 5.3.5. Although we have defined a differential highest ramification break, we have not defined a full set of differential ramification breaks, among which our highest ramification break is the largest number occurring. For the present paper, the highest ramification break is enough; for the construction of the other breaks, see [15].

6 Local semistable reduction for monomial valuations

We conclude by proving local semistable reduction for monomial valuations.

6.1 Monomial valuations

Definition 6.1.1. Let F be a finitely generated field over k. A valuation v on F over k is *monomial* (in the sense of [17, Definition 2.5.3]) if

$$\mathrm{rank}(v) = 1, \qquad \mathrm{ratrank}(v) = \mathrm{trdeg}(F/k), \qquad \kappa_v = k.$$

Note that v is then *minimal* in the sense of [17, Definition 4.3.2]. Moreover, v is an Abyhankar valuation in the sense of [17, Definition 2.5.3], which forces the value group of v to be a a finite free $\mathbb{Z}$-module.

Proposition 6.1.2. *Let F be a finitely generated field over k, let v be a monomial valuation on F with residue field k, and let $x_1, \dots, x_n \in F$ be such that $v(x_1), \dots, v(x_n)$ freely generate the value group of v. Then the completion $\widehat{F}$ is isomorphic to the completion $k((x_1, \dots, x_n))_v$ of $k(x_1, \dots, x_n)$ under v, i.e., the set of formal sums $\sum_I a_I x^I$ with $a_I \in k$ such that for any $c \in \mathbb{R}$, there are only finitely many indices I with $v(x^I) < c$ and $a_I \neq 0$.*

Proof. (For properties of valuations used in this argument, see for instance [26, Chapter 6].) The extension $\widehat{F}$ of $k((x_1, \dots, x_n))_v$ is finitely generated and of transcendence degree 0, and hence finite. Suppose this extension is nontrivial. Since it is immediate (it changes neither the value group nor the residue field), by Ostrowski's theorem [26, Theorem 6.1.2], its degree is a power of p, as is the degree of its Galois closure. By an elementary argument with p-groups, $\widehat{F}$ contains an Artin-Schreier subextension which is also immediate.

However, any Artin-Schreier extension of $k((x_1, \dots, x_n))_v$ can be written as $z^p - z = P(x_1, \dots, x_n)$, where no monomial of P of negative degree is a p-th power. Hence one of the following is true, yielding a contradiction.

- We have $v(P) \geq 0$, in which case the extension is unramified and hence not immediate.
- We have $v(P) < 0$, and the lowest degree monomial of P has valuation not divisible by p in the value group; then the extension has strictly larger value group, so is not immediate.
- We have $v(P) < 0$, and the lowest degree monomial of P has valuation divisible by p, but its coefficient is not a p-th power in k; then the extension has strictly larger residue field, so is not immediate.

This yields the desired result. □

Since monomial valuations are Abyhankar valuations, they satisfy local uniformization; the following is a special case of [21, Theorem 1.1].

Proposition 6.1.3. *Let F be a finitely generated field over a field k, let v be a monomial valuation on F, and let Z be a finite subset of the valuation ring $\mathfrak{o}_v$. Then there exists an irreducible k-scheme of finite type X with $k(X) = F$, on which v is centered at a smooth closed point x, and a system of parameters $a_1, \dots, a_n$ of X at x such that each $z \in Z$ can be written as a unit in $\mathcal{O}_{X,x}$ times a monomial in the a_i.*

Definition 6.1.4. Let X be a smooth irreducible k-variety, and let v be a monomial valuation on $k(X)$ centered at a point $x \in X$. We say a system of parameters $a_1, \dots, a_n$ for X at x is *descriptive* for v if $v(a_1), \dots, v(a_n)$ generate $v(k(X)^*)$.

Proposition 6.1.5. *Let (X, D) be a smooth pair over an algebraically closed field k with X irreducible, and let v be a monomial valuation on $k(X)$ over k centered on X. Then there exist a smooth pair (X', D'), a birational (regular) morphism $f : X' \to X$, a point $x' \in X'$, and a system of parameters $a_1, \dots, a_n$ for X' at x', such that:*

- *$f^{-1}(D) \subseteq D'$;*
- *v is centered at x';*
- *$a_1, \ldots, a_n$ is descriptive for v;*
- *each component of D' is the zero locus of one of the a_i.*

Proof. We may as well take X to be affine. Take the set Z to contain:

(a) a set of generators of the coordinate ring $k[X]$ as a k-algebra;

(b) a sequence $t_1, \ldots, t_n$ such that $v(t_1), \ldots, v(t_n)$ freely generate $v(k(X)^*)$ as a $\mathbb{Z}$-module;

(c) some functions which cut out the components of D passing through the center of v on X.

Apply Proposition 6.1.3; if we take X' to be a sufficiently small open affine neighborhood of the center x' of v on the resulting scheme, and take D' to be the zero locus of $a_1, \ldots, a_n$, then (X', D') will form a smooth pair. By (a), there will be a birational regular map $f : X' \to X$. By (b), $v(a_1), \ldots, v(a_n)$ generate $v(k(X)^*)$ as a $\mathbb{Z}$-module. By (c), we can force $f^{-1}(D) \subseteq D'$ by possibly shrinking X'. This yields the desired result. □

Proposition 6.1.6. *Let (X, D) be a smooth pair over an algebraically closed field k with X irreducible, let v be a monomial valuation on $k(X)$ centered at a point $x \in D$, let F be a finite Galois extension of $k(X)$, and let w be an extension of v to F. For (X', D') a toroidal blowup of $(U, U \cap D)$ for some open neighborhood U of x in X, write $f : Y' \to X'$ for the normalization of X' in F. Then it is possible to choose (X', D') such that $(Y', f^{-1}(D'))$ is a smooth pair and w is centered on Y'.*

Proof. We may assume without loss of generality that x is the intersection of all of the components of D. Let y' denote the center of w on Y'.

Note that the conclusion implies that in a neighborhood of y', the pullback of D' to Y' as a Cartier divisor is a $\mathbb{Z}$-linear combination of the components of $f^{-1}(D')$. Consequently, if F' is an intermediate field between $k(X)$ and F, we can prove the claim by first passing from $k(X)$ to F' and then from F' to F: the point is that in the second step, the toroidal blowup on the middle variety in the tower is induced by a toroidal blowup on the bottom variety.

We can write $F/k(X)$ as a tower $F/T/U/k(X)$, where $U/k(X)$ is unramified at v, T/U is totally tamely ramified at v, and F/T is a p-power extension for $p = \mathrm{char}(k)$ (or the trivial extension if $\mathrm{char}(k) = 0$). Moreover, by elementary group theory, F/T can be written as a tower of $\mathbb{Z}/p\mathbb{Z}$-extensions. We may thus reduce to the cases where $F/k(X)$ is unramified, tamely ramified, or an Artin-Schreier extension.

There is nothing to check in the unramified case. In the tamely ramified case, the morphism $Y' \to X'$ is toroidal, so $(Y', f^{-1}(D'))$ is automatically toroidal; it thus suffices to perform toroidal resolution of singularities [20] upstairs, as again we can mimic the toroidal

blowups downstairs. In the Artin-Schreier case, we have $F = k(X)[z]/(z^p - z - h)$ for some $h \in k(X)$ with $v(h) < 0$. By Proposition 6.1.3 (or a direct calculation), we can choose the blowup (X', D') so that at $x' = f(y')$, h^{-1} becomes a unit in $\mathcal{O}_{X',x'}$ times a product of powers of local parameters of components of D' at x'. Then $(Y', f^{-1}(D'))$ is toroidal, so again toroidal resolution of singularities yields the claim. □

Remark 6.1.7. Beware that in Proposition 6.1.6, the morphism $Y' \to X'$ is in general not toroidal when $\mathrm{char}(k) = p > 0$. This is already true for curves: consider the covering

$$\operatorname{Spec} k[x,t]/(t - x^p - x^{p+1}) \to \operatorname{Spec} k[x].$$

6.2 The contagion of unipotence

Proposition 6.2.1. *Let S be the set of n-tuples $(\rho^{x_1}, \dots, \rho^{x_{n-1}}, \rho)$ for ρ in some interval $(\epsilon, 1)$ and $x = (x_1, \dots, x_{n-1})$ in some rational polyhedral subset U of $\mathbb{R}^{n-1}$. Let q be a power of p. Let $\mathcal{E}$ be a ∇-module on $A_K(S)$ equipped with an isomorphism $\sigma^*\mathcal{E} \cong \mathcal{E}$ on $A_K(S^{1/q})$ for some map $\sigma : A_K(S^{1/q}) \to A_K(S)$ obtained by composing the toric map f_{qI_n} with a q-power Frobenius lift on K. Suppose $y \in U$ is such that $1, y_1, \dots, y_{n-1}$ are linearly independent over $\mathbb{Q}$, and*

$$T(\mathcal{E}, (\rho^{y_1}, \dots, \rho^{y_{n-1}}, \rho)) = 1 \qquad \text{for } \rho \in (\epsilon, 1) \text{ sufficiently close to } 1.$$

Then there exists a neighborhood V of y in U such that for $x \in V$, $T(\mathcal{E}, (\rho^{x_1}, \dots, \rho^{x_{n-1}}, \rho)) = 1$ for $\rho \in (\epsilon, 1)$ sufficiently close to 1.

Proof. By Theorem 4.4.7 (applicable because of Remark 4.4.8), for each $x \in U \cap \mathbb{Q}^{n-1}$, there exists $f(x) \geq 0$ with

$$f(x) \in \frac{1}{\mathrm{rank}(\mathcal{E})!}(\mathbb{Z} + x_1\mathbb{Z} + \cdots + x_{n-1}\mathbb{Z}).$$

such that

$$T(\mathcal{E}, (\rho^{x_1}, \dots, \rho^{x_{n-1}}, \rho)) = \rho^{f(x)} \qquad \text{for } \rho \in (\epsilon, 1) \text{ sufficiently close to } 1.$$

Moreover, $f(x)$ is convex by Proposition 5.3.2. Thus we may apply Theorem 2.3.2 to deduce that $\mathrm{rank}(\mathcal{E})!f$ is internally integral polyhedral.

The boundaries between domains of affinity of f all lie on rational hyperplanes, whereas y lies on no such hyperplanes because $1, y_1, \dots, y_{n-1}$ are linearly independent over $\mathbb{Q}$. Hence y lies in the interior of some domain of affinity. In that domain, there exist $a_1, \dots, a_{n-1}, b \in \mathbb{Z}$ such that

$$\mathrm{rank}(\mathcal{E})!f(x) = a_1x_1 + \cdots + a_{n-1}x_{n-1} + b.$$

Since $f(y) = 0$ and $1, y_1, \dots, y_{n-1}$ are linearly independent over $\mathbb{Q}$, we must have $a_1 = \cdots = a_{n-1} = b = 0$, that is, $f(x) = 0$ identically in an open neighborhood of y, as desired. □

6.3 F-isocrystals near a monomial valuation

We are now ready to prove our first instances of local semistable reduction at a minimal valuation on a variety of dimension greater than 1. (The theorem also applies for X of dimension 1, but in that case one can simply apply the usual p-adic local monodromy theorem for the same effect.)

Theorem 6.3.1. *Let X be a smooth irreducible k-scheme, let $\overline{X}$ be a partial compactification of X, and let $\mathcal{E}$ be an F-isocrystal on X overconvergent along $\overline{X} \setminus X$. Then $\mathcal{E}$ admits local semistable reduction at any monomial valuation on $k(X)$ centered on $\overline{X}$.*

Proof. We may assume k is algebraically closed thanks to [17, Proposition 3.2.6]. Let v be a monomial valuation on $k(X)$. By Proposition 6.1.5, there is a smooth pair (Y, D) containing an open dense subscheme of X, such that v is centered at an intersection of components of D, and the valuations of some system of parameters $t_1, \dots, t_n$ at that point freely generate $v(k(X)^*)$.

Put $y_i = v(t_i)/v(t_n)$ for $i = 1, \dots, n-1$; we can then realize $\mathcal{E}$ as a ∇-module on $A_K(S)$ for some set S containing $(\rho^{x_1}, \dots, \rho^{x_{n-1}}, \rho)$ for ρ in some interval $(\epsilon, 1)$ and $x = (x_1, \dots, x_{n-1})$ in some neighborhood of y in $\mathbb{R}^{n-1}$. Moreover, $\mathcal{E}$ admits a Frobenius action for a Frobenius lift on $A_K(S)$ given by composing a q-power Frobenius lift on K with the toric map f_{qI_n}. Take $\lambda = (y_1, \dots, y_{n-1}, 1)$ and form the (F, ∇)-module $M_{\mathcal{E}}$ over $\mathcal{R}^\lambda$ corresponding to $\mathcal{E}$. If $M_{\mathcal{E}}$ is unipotent, we may apply Proposition 6.2.1 to deduce that for x in a possibly smaller neighborhood of y, $T(\mathcal{E}, (\rho^{x_1}, \dots, \rho^{x_{n-1}}, \rho)) = 1$ for ρ sufficiently close to 1.

This means (by virtue of Proposition 5.3.4 applied after a toric coordinate change) that by passing to a suitable toroidal blowup in the sense of [20], we can obtain another smooth pair (Y', D') such that v is centered at the intersection of n components of D', and $\mathcal{E}$ becomes unipotent along each of those components after making a suitable tamely ramified cover. (For instance, it suffices to perform a blowup corresponding to a barycentric subdivision sufficiently many times.) If we take m sufficiently divisible and prime to p, then pass to a cover that is tamely ramified of degree m along each of the n components of D', we get a smooth pair (Y'', D'') on which v is centered at an intersection of components of D'', along each of which $\mathcal{E}$ is unipotent. By [16, Theorem 6.4.5], $\mathcal{E}$ extends to a log-isocrystal with nilpotent residues on (Y'', D'').

If $M_{\mathcal{E}}$ is not unipotent, we apply Theorem 5.1.8 (to produce a good finite cover) and Proposition 6.1.6 (to toroidalize) to deduce that after passing up to a suitable quasi-resolution, we get into the situation where $M_{\mathcal{E}}$ is indeed unipotent. This yields local semistable reduction at v, as desired. $\square$

By virtue of earlier work, we obtain the same conclusion more generally for Abhyankar valuations.

Corollary 6.3.2. *Let X be a smooth irreducible k-scheme, let $\overline{X}$ be a partial compactification of X, let $\mathcal{E}$ be an F-isocrystal on X overconvergent along $\overline{X} \setminus X$, and let v be any Abhyankar valuation on $k(X)$ centered on $\overline{X}$. Then $\mathcal{E}$ admits local semistable reduction at v.*

Proof. This follows from Theorem 6.3.1 as in the proofs of [17, Proposition 4.2.4 and Theorem 4.3.4]. □

A Some examples

In this appendix, we make good on two promises of examples to illustrate aspects of the semistable reduction problem.

A.1 Finite covers are not enough

The following example illustrates that one cannot necessarily render unipotent the local monodromy of an overconvergent F-isocrystal by pulling back along a finite cover instead of an alteration, as alluded to in the introduction of [16].

Example A.1.1. Let $\mathcal{F}$ be the pullback along the map $t \mapsto t^{-1}$ of the Bessel isocrystal on $\mathbb{G}_m$, as defined in [30, Example 6.2.6]. Then there exists a finite flat morphism $f : X \to \mathbb{P}^1_k$ such that $f^*\mathcal{F}$ extends to a convergent log-isocrystal $\mathcal{F}_1$ on $(X, f^{-1}(\{0, \infty\}))$, and the Frobenius slopes of $\mathcal{F}_1$ at a closed point $x \in X$ equal $1/2, 1/2$ if $f(x) = \infty$ and $0, 1$ otherwise.

Let $\pi_1, \pi_2 : \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k$ denote the canonical projections, and put $\mathcal{E} = \pi_1^*\mathcal{F} \otimes \pi_2^*\mathcal{F}$. Based on the properties of $\mathcal{F}$, we know that there exists an alteration $f_1 : X_1 \to \mathbb{P}^1_k \times \mathbb{P}^1_k$ such that $f_1^*\mathcal{E}$ extends to a convergent log-isocrystal $\mathcal{E}_1$ on X_1 for some log structure. Moreover, for one such alteration, the Frobenius slopes of $\mathcal{E}_1$ at a closed point $x \in X_1$ equal

$$\begin{cases} 1, 1, 1, 1 & f_1(x) = (\infty, \infty) \\ 1/2, 1/2, 3/2, 3/2 & f_1(x) \in (\{\infty\} \times \mathbb{A}^1_k) \cup (\mathbb{A}^1_k \times \{\infty\}) \\ 0, 1, 1, 2 & f_1(x) \in \mathbb{A}^1_k \times \mathbb{A}^1_k; \end{cases} \tag{A.1.1.1}$$

it follows that the same holds for *any* such alteration. (Given a second such alteration $f_2 : X_2 \to \mathbb{P}^1_k \times \mathbb{P}^1_k$, we can construct a third alteration $f_3 : X_3 \to \mathbb{P}^1_k \times \mathbb{P}^1_k$ factoring through both f_1 and f_2, then transfer the information about the Frobenius slopes from X_1 to X_3 to X_2.)

We now wish to argue that there cannot exist a finite morphism $f : X \to \mathbb{P}^2_k$ such that $f^*\mathcal{E}$ extends to a convergent log-isocrystal on X for some log structure. To see this, we may reduce to the case where f is Galois (by replacing the cover by its normal closure), in which case the Frobenius slopes of the extension of $f^*\mathcal{E}$ at a point $x \in X$ depend only on the projection $f(x)$.

Let P be the closure of the graph of a rational map $\mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ identifying $\mathbb{A}^1_k \times \mathbb{A}^1_k$ with $\mathbb{A}^2_k$. Put $Y = X \times_{\mathbb{P}^2_k} P$, so that base change induces a finite morphism $f : Y \to P$, and let f_1 denote the composition $Y \xrightarrow{f} P \to \mathbb{P}^1_k \times \mathbb{P}^1_k$. Then the above analysis shows that the Frobenius slopes of the extension of $f_1^*\mathcal{E}$ at a point $y \in Y$ depend only on $f_1(y)$.

However, this yields a contradiction as follows. Each of the three components of $Z = P \setminus \mathbb{A}^2_k$ is contracted by one of the projections $P \to \mathbb{P}^1_k \times \mathbb{P}^1_k$ or $P \to \mathbb{P}^2_k$. Consequently,

the Frobenius slopes must be constant along each component; since Z is connected, the slopes must be constant along all of $f_1^{-1}(Z)$. However, this contradicts the explicit formula (A.1.1.1).

Remark A.1.2. This example is not meant to suggest that one is compelled to blow up in the locus where the isocrystal is already defined. Indeed, it is entirely possible that one can always use an alteration which is finite étale over that locus; however, even if one had as strong a form of resolution of singularities in positive characteristic as desired, it is not clear how to use the valuation-theoretic approach to prove this refined form of semistable reduction.

A.2 Extra monodromy on exceptional divisors

The following example illustrates that one cannot necessarily render unipotent the local monodromy of an overconvergent F-isocrystal by doing so only for the divisors in a specified good compactification of the locus of definition, as alluded to in the introduction of this paper.

Example A.2.1. Consider an affine plane $\mathbb{A}^2_k$ with coordinates x, y, embed it into a projective plane $\mathbb{P}^2_k$, and let X be the complement of the line $y = 0$ in $\mathbb{P}^2_k$. View $P(x, y, z) = yz^{p^2} - x^{p-1}z^p + z$ as a polynomial in $k(x, y)[z]$. One checks that the extension $k(x, y)[z]/(P)$ of $k(x, y) = k(X)$ defines a finite étale cover $f : Y \to X$. Let $\mathcal{E}$ be the overconvergent F-isocrystal $f_*\mathcal{O}_Y$ on X. We consider twisted polynomials again as in [24], but for the Frobenius automorphism instead of for a derivation. Over the y-adic completion $k(x)((y))$ of $k(x, y)$, we can factor the twisted polynomial $Q = yF^2 - x^{p-1}F + 1$ as $(yF - c)(F - 1/c)$ for some $c \equiv x^{p-1} \pmod{y}$; in particular, c has a $(p-1)$-st root in $k(x)((y))$. We may thus split P over an Artin-Schreier extension of $k(x)((y))$; by Krasner's lemma, we can realize this as the completion of a degree p extension of $k(x, y)$.

This means that we can construct a finite flat morphism $g : Y_1 \to \mathbb{P}^2_k$ of degree p such that $g^*\mathcal{E}$ has constant local monodromy along each component of the proper transform of the line $y = 0$. However, if we blow up at $x = y = 0$ and complete the function field along the resulting exceptional divisor, we obtain $k(x/y)((y))$, over which Q remains irreducible. Consequently, $g^*\mathcal{E}$ cannot have constant local monodromy along the proper transform of the exceptional divisor.

Remark A.2.2. In Example A.2.1, the overconvergent F-isocrystal $\mathcal{E}$ is unit-root because it is a pushforward of the unit-root isocrystal $\mathcal{O}_Y$. Hence one can recover semistable reduction for $\mathcal{E}$ using results of Tsuzuki [31]. The method of proof follows the model one would use in the ℓ-adic setting: convert $\mathcal{E}$ into a p-adic representation of the étale fundamental group of X, choose a stable lattice, and pick a finite étale cover of X that trivializes a suitable quotient of the lattice. Unfortunately, without a unit-root condition, one has no useful functor from isocrystals to Galois representations; the compactness of the Riemann-Zariski space serves as a replacement for this construction.

References

[1] Y. Amice, *Les nombres p-adiques*, Presses Universitaires de France, Paris, 1975.

[2] Y. André, Filtrations de type Hasse-Arf et monodromie p-adique, *Invent. Math.* **148** (2002), 285–317.

[3] V.G. Berkovich, *Spectral theory and analytic geometry over non-archimedean fields* (translated by N.I. Koblitz), Math. Surveys and Monographs 33, Amer. Math. Soc., Providence, 1990.

[4] G. Christol and B. Dwork, Modules différentiels sur des couronnes, *Ann. Inst. Fourier (Grenoble)* **44** (1994), 663–701.

[5] G. Christol and Z. Mebkhout, Sur le théorème de l'indice des équations différentielles p-adiques. III, *Annals of Math.* **151** (2000), 385–457.

[6] G. Christol and P. Robba, *Équations différentielles p-adiques, Applications aux sommes exponentielles*, Actualités Mathématiques, Hermann, Paris, 1994.

[7] A.J. de Jong, Smoothness, semi-stability and alterations, *Inst. Hautes Études Sci. Publ. Math.* **83** (1996), 51–93.

[8] B. Dwork, G. Gerotto, and F.J. Sullivan, *An introduction to G-functions*, Annals of Math. Studies 133, Princeton Univ. Press, Princeton, 1994.

[9] B. Dwork and P. Robba, On ordinary linear p-adic differential equations, *Trans. Amer. Math. Soc.* **231** (1977), 1–46.

[10] S. Fujishige, *Submodular functions and optimization*, second edition, Annals of Discrete Math. 58, Elsevier, Amsterdam, 2005.

[11] K.S. Kedlaya, A p-adic local monodromy theorem, *Annals of Math.* **160** (2004), 93–184.

[12] K.S. Kedlaya, Local monodromy for p-adic differential equations: an overview, *Intl. J. of Number Theory* **1** (2005), 109–154.

[13] K.S. Kedlaya, Slope filtrations revisited, *Doc. Math.* **10** (2005), 447–525.

[14] K.S. Kedlaya, The p-adic local monodromy theorem for fake annuli, *Rend. Sem. Mat. Padova* **118** (2007), 101–146.

[15] K.S. Kedlaya, Swan conductors for p-adic differential modules, I: A local construction, *Alg. and Number Theory* **1** (2007), 269–300.

[16] K.S. Kedlaya, Semistable reduction for overconvergent F-isocrystals, I: Unipotence and logarithmic extensions, *Compos. Math.* **143** (2007), 1164–1212.

[17] K.S. Kedlaya, Semistable reduction for overconvergent F-isocrystals, II: A valuation-theoretic approach, *Compos. Math.* **144** (2008), 657–672.

[18] K.S. Kedlaya, Slope filtrations for relative Frobenius, *Astérisque*, to appear; arXiv preprint `math.NT/0609272v2` (2007).

[19] K.S. Kedlaya, p-adic differential equations (version of 8 Feb 08), preprint available at `http://math.mit.edu/~kedlaya/papers/`.

[20] G. Kempf, F.F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Math. 339, Springer-Verlag, Berlin, 1973.

[21] H. Knaf and F.-V. Kuhlmann, Abhyankar places admit local uniformization in any characteristic, *Ann. Scient. Éc. Norm. Sup.* **38** (2005), 833–846.

[22] F.-V. Kuhlmann, Places of algebraic function fields in arbitrary characteristic, *Adv. Math.* **188** (2004), 399–424.

[23] Z. Mebkhout, Analogue p-adique du Théorème de Turrittin et le Théorème de la monodromie p-adique, *Invent. Math.* **148** (2002), 319–351.

[24] O. Ore, Theory of non-commutative polynomials, *Ann. Math.* **34** (1933), 480–508.

[25] G. Pólya and G. Szegő, *Problems and theorems in analysis*, part I, reprint of the 1978 edition, Springer-Verlag, Berlin, 1998.

[26] P. Ribenboim, *The theory of classical valuations*, Springer-Verlag, New York, 1999.

[27] P. Robba, Lemmes de Hensel pour les opérateurs différentiels. Application a la réduction formelle des équations différentielles, *Enseign. Math.* (2) **26** (1980), 279–311.

[28] R.T. Rockafellar, *Convex analysis*, Princeton Univ. Press, Princeton, 1970.

[29] C. Sabbah, Équations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension 2, *Astérisque* **263** (2000).

[30] N. Tsuzuki, Slope filtration of quasi-unipotent overconvergent F-isocrystals, *Ann. Inst. Fourier (Grenoble)* **48** (1998), 379–412.

[31] N. Tsuzuki, Morphisms of F-isocrystals and the finite monodromy theorem for unit-root F-isocrystals, *Duke Math. J.* **111** (2002), 385–418.

[32] P.T. Young, Radii of convergence and index for p-adic differential operators, *Trans. Amer. Math. Soc.* **333** (1992), 769–785.